‘Constraint consistency’ at all orders in Cosmological perturbation theory

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Abstract. We study the equivalence of two — order-by-order Einstein's equation and Reduced action — approaches to cosmological perturbation theory at all orders for different models of inflation. We point out a crucial consistency check which we refer to as ‘Constraint consistency’ condition that needs to be satisfied in order for the two approaches to lead to identical single variable equation of motion. The method we propose here is quick and efficient to check the consistency for any model including modified gravity models. Our analysis points out an important feature which is crucial for inflationary model building i.e., all ‘constraint’ inconsistent models have higher order Ostrogradsky's instabilities but the reverse is not true. In other words, one can have models with constraint Lapse function and Shift vector, though it may have Ostrogradsky's instabilities. We also obtain single variable equation for non-canonical scalar field in the limit of power-law inflation for the second-order perturbed variables.

1 Introduction

Inflation has now become an integral part of the standard model, that can eliminate cosmological initial value problems, explain homogeneities as well as inhomogeneities and observation of anisotropic Cosmic Microwave Background Radiation (CMBR)[1]. It is a period of accelerated expansion in the very early universe and it occurs around 10^{14} GeV which is much remote in time compared to the terrestrial experiments. Inflationary cosmology has two key theoretical aspects. One is the approximation schemes employed in solving gravity equations. The other is the inflationary model building inspired by particle physics or a fundamental theory of Quantum gravity. The problem with any theory of gravity is that it is typically highly non-linear, so one has to rely on approximation schemes to match the observations. Primarily there exist two formalisms to deal with the non-linear equations:

- The separate universe approximations[2–4] with either gradient expansion theory or ΔN formalism.[5–11]
- Gauge invariant cosmological perturbation theory[12–34].

The temperature fluctuations as observed in CMB is $\sim 10^{-5}$, hence it is consistent to use order-by-order perturbation theory to match with observations[12, 35, 36]. In the first order, one assumes that the perturbed fields are linear. This implies that the 3-point and higher order correlation functions are zero. In the second order, the interactions of the first order need to be included, hence, leading to non-zero 3-point functions. Also it is widely believed that the detection of these 3-point correlation functions can reduce the field space of inflationary models[14, 17, 37].

With respect to inflationary model building, the proposed theories are primarily preferred through simplicity. In the case of the canonical scalar field, the simplest, 60 e-foldings of inflation require the potential to be flat, which is in contradiction with particle physics models[38, 39]. Non-canonical scalar field model[40–42] removes the dependence of the potential, however it leads to time dependence of the speed of perturbations and makes it difficult to be compared with CMB observations[43]. In order to seek more generalized fields, scalar fields with higher time derivatives in action are considered[44–47]. Beside these, modified gravity models, specifically $f(R)$ lead to accelerated expansion in the early universe.

There are two mathematical procedures that are currently used in the literature to study gauge invariant cosmological perturbation theory: Hamiltonian formulation(ADM formulation)[34] and Lagrangian formulation[12–33]. Since gravity and matter are coupled to each other, one can write the full action and vary the action with respect to metric and matter fields to obtain general equations of motion (e.g., Einstein's equation in General relativity). Those equations can be expanded in terms of perturbed variables (metric and field variables) and one can write down equations in the perturbation theory[48]. In the action formalism[49], the action is expanded to the required perturbed order in terms of the perturbed variables (metric and field variables) and then varied the action with respect to these variables. For example, to obtain perturbed equations in the first order, we need to expand the action to second order and vary the action with respect to the first order perturbed variables. This formalism can be extended to obtain perturbed equations of motion up to any order. In the reduced action formalism, constraint variables are replaced in the action using constraint equations so that we can rewrite the action only in terms of dynamical variables.

Since the matter fields (Non-canonical & Galilean scalar field model) and Gravity are highly non-linear, it is not clear whether the two approaches, i.e., Einstein's equations writing

in order-by-order perturbation theory and action/reduced action formalism, lead to the same equations of motion. In Ref.[50], it was shown that when the metric perturbations are frozen then the two approaches do not, in general, lead to the same expressions. In this work we address the issue by including the metric perturbations in the theory.

In the next section, we study higher order cosmological perturbation theory for a single scalar field minimally coupled to gravity and show the equivalence of the two approaches at all orders. We point out a crucial and novel consistency check which we refer to as 'constraint consistency' condition that needs to be verified. We also show that this provides a fast and efficient way to check the consistency and apply it to minimally coupled non-canonical scalar field.

In section 3, we apply the 'constraint consistency' condition to many inflationary models that are proposed in the literature. First we check the theory with higher derivative Lagrangian models minimally coupled to gravity. Then we extend the procedure to other different types of models like modified gravity models and modified gravity with higher order matter Lagrangian. Appendix A contains some of the derived expressions used in section 2 and in Appendix B, we obtain a single variable equation of motion for non-canonical scalar fields in terms of second order perturbed variables.

In this work, the number of space-time dimensions is 4 and the metric signature we use is $[-,+,+,+]$, $\kappa = 8\pi G$, $c = 1$.

2 Consistency of Higher order perturbations in two different approaches

The action for gravity sourced by a single, non-minimally coupled scalar field (φ) is,

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} + \mathcal{L}_m\right) \tag{2.1}$$

where

$$\mathcal{L}_m = P(X,\varphi) + G(X,\varphi)\Box\varphi, \qquad X \equiv \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\mu\varphi, \qquad \Box \equiv -\frac{1}{\sqrt{-g}}\partial_\mu\{\sqrt{-g}g^{\mu\nu}\partial_\nu\} \tag{2.2}$$

is the Lagrangian for the Galilean field which is the most general scalar field model leading to second order equations of motion. Varying the action with respect to metric gives Einstein's equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R = \kappa\, T_{\mu\nu}, \tag{2.3}$$

where the stress tensor $T_{\mu\nu}$ is,

$$\begin{aligned} T_{\mu\nu} = g_{\mu\nu}\{P + G_X g^{\alpha\beta}\partial_\alpha X\partial_\beta\varphi + G_\varphi g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\} \\ - \{P_X + 2G_\varphi + G_X\Box\varphi\}\partial_\mu\varphi\partial_\nu\varphi - 2G_X\partial_\mu X\partial_\nu\varphi \end{aligned} \tag{2.4}$$

For simplicity and to obtain the physical features, we consider only single scalar field theory minimally coupled to the gravity. In Section 3.4, we look at modified gravity models. Variation of the action (2.1) with respect to the scalar field 'φ' leads to the following equation of motion,

$$\begin{aligned} &\{2G_\varphi - 2XG_{X\varphi} + P_X\}\Box\varphi - \{P_{XX} + 2G_{X\varphi}\}\partial_\mu\varphi\partial^\mu\varphi - 2X\{G_\varphi + P_{X\varphi}\} + P_\varphi \\ &- G_X\{\varphi_{,\mu\nu}\varphi^{\mu\nu}_{,} - \{\Box\varphi\}^2 + R_{\mu\nu}\partial^\mu\varphi\partial^\nu\varphi\} - G_{XX}\{\partial_\mu X\partial^\mu X + \{\partial_\mu\varphi\partial^\mu X\}\Box\varphi\}\} = 0 \end{aligned} \tag{2.5}$$

As one can see, although the Lagrangian is of the form, $\mathcal{L}_m = \mathcal{L}_m(\varphi, \partial\varphi, \partial^2\varphi, t)$, i.e., it contains higher time derivatives of the scalar field but equations of motion are second order, thus does not suffer from Ostrogradsky's instability[51]. With $G(X, \varphi) = 0$ the field becomes non-canonical. Further fixing $P = -X - V(\varphi)$, where $V(\varphi)$ is the potential, the Lagrangian corresponds to canonical scalar field.

The four-dimensional line element in the ADM form is given by,

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -(N^2 - N_i N^i) d\eta^2 + 2 N_i dx^i d\eta + \gamma_{ij} dx^i dx^j , \end{aligned} \tag{2.6}$$

where $N(x^\mu)$ and $N_i(x^\mu)$ are Lapse function and Shift vector respectively, γ_{ij} is the 3-D space metric. Note that, in the case of Galilean model, $N(x^\mu)$ and $N_i(x^\mu)$ are the gauge constraints and variation of action (2.1) with respect to those lead to Hamiltonian and Momentum constraints, respectively.

Action (2.1) for the line element (2.6) takes the form,

$$\mathcal{S} = \int d^4x \sqrt{-g} \Big\{ \frac{1}{2\kappa} \left({}^{(3)}R + K_{ij} K^{ij} - K^2 \right) + \mathcal{L}_m \Big\} \tag{2.7}$$

where K_{ij} is extrinsic curvature tensor and is given by

$$\begin{aligned} K_{ij} &\equiv \frac{1}{2N} \Big[\partial_0 \gamma_{ij} - N_{i|j} - N_{j|i} \Big] \\ K &\equiv \gamma^{ij} K_{ij} \end{aligned}$$

Perturbatively expanding the metric and the scalar field about the flat FRW spacetime, we get,

$$g_{00} = -a^2 (1 + 2\epsilon\phi_1 + \epsilon^2 \phi_2 + ...) \tag{2.8}$$

$$g_{0i} \equiv N_i = a^2 (\epsilon \partial_i B_1 + \frac{1}{2} \epsilon^2 \partial_i B_2 + ...) \tag{2.9}$$

$$g_{ij} = a^2 \{ (1 - 2\epsilon\psi_1 - \epsilon^2 \psi_2 - ...) \delta^{ij} + 2\epsilon E_{1|ij} + \epsilon^2 E_{2|ij} + ... \} \tag{2.10}$$

$$\varphi = \varphi_0 + \epsilon\varphi_1 + \frac{1}{2} \epsilon^2 \varphi_2 + ... \tag{2.11}$$

where 'ϵ' denotes the order of the perturbation. Note that we have ignored the vector and tensor part of the metric perturbations. Although in the first order, the scalar, vector and tensor perturbations decouple, the three types of perturbations are coupled in higher order. We assume that the vector and tensor contributions are small and can be neglected at all orders.

To determine the dynamics at every order, we need five scalar functions (ϕ, B, ψ, E and φ) at each order. Since there are two gauge choices, one can fix two of the five scalar functions. In this work, we choose flat-slicing gauge, i.e., $\psi = 0, E = 0$ at all orders,

$$g_{00} = -a^2 (1 + 2\epsilon\phi_1 + \epsilon^2 \phi_2 + ...) \tag{2.12}$$

$$g_{0i} \equiv N_i = a^2 (\epsilon \partial_i B_1 + \frac{1}{2} \epsilon^2 \partial_i B_2 + ...) \tag{2.13}$$

$$g_{ij} = a^2 \delta^{ij} \tag{2.14}$$

$$\varphi = \varphi_0 + \epsilon\varphi_1 + \frac{1}{2} \epsilon^2 \varphi_2 + ... \tag{2.15}$$

In the next subsection, we obtain the equation of motion of the second order perturbed quantity in single variable form for non-canonical scalar field using order-by-order perturbed Einstein's equation. In subsection 2.2, we use reduced action approach. To confirm or infirm the result of Ref.[50], that the two approaches lead to different results we focus on non-canonical scalar field, i.e., setting $G(X,\varphi)=0$.

2.1 Order-by-Order Einstein's equation approach

For the background, $g_{\mu\nu}=diag(-a^2,a^2,a^2,a^2)$, equations (2.3) and (2.5) lead to,

$$-\frac{\kappa}{3}(P_X{\varphi_0'}^2+Pa^2)=\mathcal{H}^2 \tag{2.16}$$

$$-2\frac{a''}{a}+\mathcal{H}^2=\kappa P a^2 \tag{2.17}$$

$$P_X\varphi_0''-P_{XX}\varphi_0''{\varphi_0'}^2a^{-2}+P_{X\varphi}{\varphi_0'}^2+2P_X\varphi_0'\mathcal{H}+P_{XX}\mathcal{H}{\varphi_0'}^3a^{-2}+P_\varphi a^2=0 \tag{2.18}$$

Equations (2.16) and (2.17) are zeroth order 0-0 and i-j Einstein's equations where as equation (2.18) is the zeroth order equation of motion of the scalar field. Similarly, the first order 0-0, 0-i Einstein's equations and equation of motion of the perturbed scalar field are,

$$\begin{aligned}\mathcal{H}\nabla^2B_1=\frac{\kappa}{2}(&P_X\phi_1{\varphi_0'}^2+2Pa^2\phi_1+P_X\varphi_0'\varphi_1'+P_{XX}\phi_1{\varphi_0'}^4a^{-2}-P_{XX}\varphi_1'{\varphi_0'}^3a^{-2}+\\&P_{X\varphi}{\varphi_0'}^2\varphi_1+P_\varphi\varphi_1a^2)\end{aligned} \tag{2.19}$$

$$\mathcal{H}\phi_1=-\frac{\kappa}{2}P_X\varphi_0'\varphi_1 \tag{2.20}$$

$$\begin{aligned}&-P_X\varphi_1''a^2-P_{XX}\phi_1'{\varphi_0'}^3+P_{XX}\varphi_1''{\varphi_0'}^2-P_{XX\varphi}\phi_1{\varphi_0'}^4+P_{XX\varphi}\varphi_1'{\varphi_0'}^3-P_\varphi\phi_1a^4\\&-P_{\varphi\varphi}a^4\varphi_1+P_X\phi_1\varphi_0''a^2+P_X\nabla^2\varphi_1a^2+P_X\phi_1'\varphi_0'a^2-2P_X\varphi_1'\mathcal{H}a^2-4P_{XX}\phi_1\varphi_0''{\varphi_0'}^2\\&+3P_{XX}\varphi_0'\varphi_1'\varphi_0''+P_{XX\varphi}\varphi_0''{\varphi_0'}^2\varphi_1-P_{X\varphi}\varphi_0'\varphi_1'a^2-P_{X\varphi}\varphi_0''a^2\varphi_1-P_{X\varphi\varphi}{\varphi_0'}^2a^2\varphi_1\\&+2P_X\phi_1\varphi_0'\mathcal{H}a^2+P_X\varphi_0'\nabla^2B_1a^2+P_{XX}\phi_1\mathcal{H}{\varphi_0'}^3-P_{XX}\varphi_1'\mathcal{H}{\varphi_0'}^2-P_{XXX}\phi_1\mathcal{H}{\varphi_0'}^5a^{-2}\\&+P_{XXX}\phi_1\varphi_0''{\varphi_0'}^4a^{-2}+P_{XXX}\varphi_1'\mathcal{H}{\varphi_0'}^4a^{-2}-P_{XXX}\varphi_1'\varphi_0''{\varphi_0'}^3a^{-2}-P_{XX\varphi}\mathcal{H}{\varphi_0'}^3\varphi_1\\&-2P_{X\varphi}\varphi_0'\mathcal{H}\varphi_1a^2=0\end{aligned} \tag{2.21}$$

Note that, there are no ϕ_1'' and B_1'' terms in the above three equations and equations (2.19) and (2.20) are, as expected, the constraint equations corresponding to Lapse function and Shift vector. Hence, ϕ_1 and B_1 are constraints and we can eliminate them from the first order equation of motion of the scalar field (2.21). In first order, single variable equation for non-canonical scalar field in terms of Mukhanov-Sasaki variable (v) is,

$$v''-c_s^2\nabla^2v-\frac{z''}{z}v=0 \tag{2.22}$$

where,

$$v\equiv a\varphi_1,\quad z\equiv\frac{a\varphi_0'}{\mathcal{H}},\quad c_s^2\equiv\frac{P_X}{P_X+2XP_{XX}} \tag{2.23}$$

Similarly, perturbed second order 0-0 and 0-i Einstein's equations for non-canonical scalar fields at second order are,

$$
\begin{aligned}
&-4\phi_1\mathcal{H}\nabla^2B_1+\mathcal{H}\nabla^2B_2-2\delta^{ij}\mathcal{H}\partial_iB_1\partial_j\phi_1-\frac{1}{2}\nabla^2B_1\nabla^2B_1+\frac{1}{2}\delta^{ij}\delta^{kl}\partial_{ik}B_1\partial_{jl}B_1\\
&+\kappa(-\frac{1}{2}P_X\delta^{ij}\partial_iB_1\partial_jB_1{\varphi_0'}^2-P\delta^{ij}\partial_iB_1\partial_jB_1a^2-\frac{1}{2}P_X\phi_2{\varphi_0'}^2-P\phi_2a^2+2P_X\phi_1^2{\varphi_0'}^2+4P\phi_1^2a^2\\
&+2P_X\phi_1\varphi_0'\varphi_1'+\frac{7}{2}P_{XX}\phi_1^2{\varphi_0'}^4a^{-2}-5P_{XX}\phi_1\varphi_1'{\varphi_0'}^3a^{-2}+P_{X\varphi}\phi_1{\varphi_0'}^2\varphi_1-\frac{1}{2}P_X\varphi_0'\varphi_2'\\
&-\frac{1}{2}P_X{\varphi_1'}^2-\frac{1}{2}P_{XX}\phi_2{\varphi_0'}^4a^{-2}-\frac{1}{2}P_{XX}\delta^{ij}\partial_iB_1\partial_jB_1{\varphi_0'}^4a^{-2}+2P_{XX}{\varphi_0'}^2{\varphi_1'}^2a^{-2}+\frac{1}{2}P_{XX}\varphi_2'{\varphi_0'}^3a^{-2}\\
&-P_{XX}\delta^{ij}\partial_iB_1\partial_j\varphi_1{\varphi_0'}^3a^{-2}-\frac{1}{2}P_{XX}\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1{\varphi_0'}^2a^{-2}-P_{X\varphi}\varphi_0'\varphi_1'\varphi_1-\frac{1}{2}P_{X\varphi}{\varphi_0'}^2\varphi_2\\
&-\frac{1}{2}P_{XXX}\phi_1^2{\varphi_0'}^6a^{-4}+P_{XXX}\phi_1\varphi_1'{\varphi_0'}^5a^{-4}-\frac{1}{2}P_{XXX}{\varphi_0'}^4{\varphi_1'}^2a^{-4}-\frac{1}{2}P_{X\varphi\varphi}{\varphi_0'}^2\varphi_1^2-P_{XX\varphi}\phi_1{\varphi_0'}^4a^{-2}\varphi_1\\
&+P_{XX\varphi}\varphi_1'{\varphi_0'}^3a^{-2}\varphi_1-\frac{1}{2}P_X\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1-\frac{1}{2}P_\varphi\varphi_2a^2-\frac{1}{2}P_{\varphi\varphi}\varphi_1^2a^2)=0
\end{aligned}
\tag{2.24}
$$

$$
\begin{aligned}
&-2\delta^{jk}\mathcal{H}\partial_jB_1\partial_{ik}B_1+\partial_i\Phi_1\nabla^2B_1+4\phi_1\mathcal{H}\partial_i\phi_1-\mathcal{H}\partial_i\phi_2-\delta^{jk}\partial_j\phi_1\partial_{ik}B_1\\
&+\kappa(-\frac{1}{2}P_X\varphi_0'\partial_i\varphi_2-P_X\varphi_1'\partial_i\varphi_1-P_{XX}\phi_1\partial_i\varphi_1{\varphi_0'}^3a^{-2}+P_{XX}\varphi_1'\partial_i\varphi_1{\varphi_0'}^2a^{-2}-P_{X\varphi}\varphi_0'\partial_i\varphi_1\varphi_1)=0
\end{aligned}
\tag{2.25}
$$

and the equation of motion of the scalar field is,

$$
\begin{aligned}
&C_XP_X+C_{XX}P_{XX}+C_{XXX}P_{XXX}+C_{XXXX}P_{XXXX}+C_{XXX\varphi}P_{XXX\varphi}\\
&+C_{XX\varphi}P_{XX\varphi}+C_{XX\varphi\varphi}P_{XX\varphi\varphi}+C_{X\varphi}P_{X\varphi}+C_{X\varphi\varphi}P_{X\varphi\varphi}+C_{X\varphi\varphi\varphi}P_{X\varphi\varphi\varphi}\\
&+C_\varphi P_\varphi+C_{\varphi\varphi}P_{\varphi\varphi}+C_{\varphi\varphi\varphi}P_{\varphi\varphi\varphi}=0
\end{aligned}
\tag{2.26}
$$

where $C_X, C_{XX}, ...$ are all second order perturbed quantities and $P_X, P_{XX}, P_{X\varphi}, ...$ are background quantities. The explicit form of C's are given in Appendix A.

It is important to note that the second order equations also do not contain ϕ_2'' and/or B_2''. Hence one can obtain a single variable equation of motion of non-canonical scalar field at second order. Malik et al obtained the single variable equation of motion in second order for canonical scalar field[52]. Also note that 'φ' evaluated in the flat-slicing gauge is a gauge invariant quantity and directly related to comoving curvature perturbation $\mathcal{R}$/curvature perturbation on uniform-density hypersurfaces ζ [53].

2.2 Reduced Action approach

In the reduced action approach, which is now a popular way to calculate non-gaussianity, one perturbs the field variables $(g_{\mu\nu}, \varphi)$ in the action and expands the action to the required order. In other words, one assumes a priori the form of the metric and the matter variables in the lowest order and expands order-by-order. For instance, in the case of FRW background, the action (2.7) becomes,

$$
{}^{(0)}\mathcal{S}^{NC}=\int d^4x\left(Pa^4-3\frac{1}{\kappa}{a'}^2\right)
\tag{2.27}
$$

Varying the above action with respect to metric variable $a(\eta)$ and $\varphi_0(\eta)$ leads to the equations (2.17) and (2.18). Note that, as expected, these two equations are independent of

each other since $a(\eta)$ and $\varphi_0(\eta)$ are dynamical variables. To obtain first order (in ϵ) equations, one expands action (2.7) upto second order of ϵ. *In general, varying the n^{th} order action with respect to m^{th} order perturbed variables leads to $(n-m)^{th}$ order perturbed equations. It may be worth noting that a given order equations of motion can be obtained in several ways, e.g., varying first order action with respect to first order variables leads to zeroth order equations of motion.*

Expanding the action (2.7) to the second order, only in terms of first order variables (φ_1, ϕ_1, B_1), we get

$$\begin{aligned}
{}^{(2)}\mathcal{S}^{NC} = \int d^4x \Big(& \frac{1}{2} P_X \delta^{ij} \partial_i B_1 \partial_j B_1 {\varphi_0'}^2 a^2 - P_X \phi_1^2 {\varphi_0'}^2 a^2 + P_X \phi_1 \varphi_0' \varphi_1' a^2 - \frac{1}{2} P_X {\varphi_1'}^2 a^2 + \\
& P_X \delta^{ij} \varphi_0' \partial_i B_1 \partial_j \varphi_1 a^2 + \frac{1}{2} P_X \delta^{ij} \partial_i \varphi_1 \partial_j \varphi_1 a^2 + \frac{1}{2} P_{XX} \phi_1^2 {\varphi_0'}^4 - P_{XX} \phi_1 \varphi_1' {\varphi_0'}^3 + \frac{1}{2} P_{XX} {\varphi_0'}^2 {\varphi_1'}^2 + \\
& \frac{1}{2} P_{\varphi\varphi} \varphi_1^2 a^4 + P_{X\varphi} \phi_1 {\varphi_0'}^2 a^2 \varphi_1 - P_{X\varphi} \varphi_0' \varphi_1' a^2 \varphi_1 + P_\varphi \phi_1 a^4 \varphi_1 + \frac{1}{2} P \delta^{ij} \partial_i B_1 \partial_j B_1 a^4 - \frac{1}{2} P \phi_1^2 a^4 - \\
& 2\phi_1 \delta^{ij} \frac{1}{\kappa} a' \partial_{ij} B_1 a + \frac{3}{2} \delta^{ij} \frac{1}{\kappa} \partial_i B_1 \partial_j B_1 {a'}^2 - \frac{9}{2} \frac{1}{\kappa} \phi_1^2 {a'}^2 + \frac{1}{2} \delta^{ij} \delta^{kl} \frac{1}{\kappa} \partial_{ik} B_1 \partial_{jl} B_1 a^2 - \\
& \frac{1}{2} \delta^{ij} \delta^{kl} \frac{1}{\kappa} \partial_{ij} B_1 \partial_{kl} B_1 a^2 \Big)
\end{aligned} \tag{2.28}$$

After integrating by-parts, and dropping off boundary terms, we get,

$$\begin{aligned}
{}^{(2)}\mathcal{S}^{NC} = \int d^4x \Big(& \frac{1}{2} P_X \delta^{ij} \partial_i B_1 \partial_j B_1 {\varphi_0'}^2 a^2 - P_X \phi_1^2 {\varphi_0'}^2 a^2 + P_X \phi_1 \varphi_0' \varphi_1' a^2 - \frac{1}{2} P_X {\varphi_1'}^2 a^2 + \\
& P_X \delta^{ij} \varphi_0' \partial_i B_1 \partial_j \varphi_1 a^2 + \frac{1}{2} P_X \delta^{ij} \partial_i \varphi_1 \partial_j \varphi_1 a^2 + \frac{1}{2} P_{XX} \phi_1^2 {\varphi_0'}^4 - P_{XX} \phi_1 \varphi_1' {\varphi_0'}^3 + \frac{1}{2} P_{XX} {\varphi_0'}^2 {\varphi_1'}^2 + \\
& \frac{1}{2} P_{\varphi\varphi} \varphi_1^2 a^4 + P_{X\varphi} \phi_1 {\varphi_0'}^2 a^2 \varphi_1 - P_{X\varphi} \varphi_0' \varphi_1' a^2 \varphi_1 + P_\varphi \phi_1 a^4 \varphi_1 + \frac{1}{2} P \delta^{ij} \partial_i B_1 \partial_j B_1 a^4 - \\
& \frac{1}{2} P \phi_1^2 a^4 - 2\phi_1 \delta^{ij} \frac{1}{\kappa} a' \partial_{ij} B_1 a + \frac{3}{2} \delta^{ij} \frac{1}{\kappa} \partial_i B_1 \partial_j B_1 {a'}^2 - \frac{9}{2} \frac{1}{\kappa} \phi_1^2 {a'}^2 \Big)
\end{aligned} \tag{2.29}$$

Varying action (2.29) with respect to φ_1, we obtain first order equation of motion of the scalar field same as (2.21). Similarly, varying action with respect to ϕ_1 and B_1 gives same equations as (2.19) and (2.20) respectively, i.e.,

$$\begin{aligned}
\left\{ \frac{\delta S_2}{\delta \varphi_1} \right\}_{\phi_1, B_1} &\equiv 1^{st} \ \textit{order Equation of motion of the scalar field} \\
\left\{ \frac{\delta S_2}{\delta \phi_1} \right\}_{\varphi_1, B_1} &\equiv 1^{st} \ \textit{order Hamiltonian constraint} \\
\left\{ \frac{\delta S_2}{\delta B_1} \right\}_{\varphi_1, \phi_1} &\equiv 1^{st} \ \textit{order Momentum constraint}
\end{aligned}$$

Similarly, we can expand (2.7) upto fourth order by expanding the field variables (φ_2, ϕ_2, B_2) and vary the action with respect to second order perturbed field variables to

obtain second order equations. Fourth order action containing only φ_2 terms are,

$$
\begin{aligned}
{}^{(4)}\mathcal{S}_{\varphi_2}^{NC} = \int d^4x \Big(& \frac{1}{4}P_X\phi_2\varphi_0'\varphi_2'a^2 + \frac{1}{4}P_X\delta^{ij}\varphi_0'\varphi_2'\partial_iB_1\partial_jB_1a^2 - \frac{3}{4}P_X\varphi_0'\varphi_2'\phi_1^2a^2 + \frac{1}{2}P_X\phi_1\varphi_1'\varphi_2'a^2 - \frac{1}{8}P_X{\varphi_2'}^2a^2 + \\
& \frac{1}{4}P_X\delta^{ij}\varphi_0'\partial_iB_2\partial_j\varphi_2a^2 - \frac{1}{2}P_X\phi_1\delta^{ij}\varphi_0'\partial_iB_1\partial_j\varphi_2a^2 + \frac{1}{2}P_X\delta^{ij}\varphi_1'\partial_iB_1\partial_j\varphi_2a^2 + \frac{1}{2}P_X\delta^{ij}\varphi_2'\partial_iB_1\partial_j\varphi_1a^2 + \\
& \frac{1}{8}P_X\delta^{ij}\partial_i\varphi_2\partial_j\varphi_2a^2 + \frac{3}{2}P_{XX}\varphi_2'\phi_1^2{\varphi_0'}^3 - 2P_{XX}\phi_1\varphi_1'\varphi_2'{\varphi_0'}^2 + \frac{1}{2}P_{XX}\phi_1\delta^{ij}\partial_iB_1\partial_j\varphi_2{\varphi_0'}^3 + \frac{1}{2}P_{XX}\phi_1\delta^{ij}\partial_i\varphi_1\partial_j\varphi_2{\varphi_0'}^2 - \\
& \frac{1}{4}P_{XX}\phi_2\varphi_2'{\varphi_0'}^3 - \frac{1}{4}P_{XX}\delta^{ij}\varphi_2'\partial_iB_1\partial_jB_1{\varphi_0'}^3 + \frac{3}{4}P_{XX}\varphi_0'\varphi_2'{\varphi_1'}^2 - \frac{1}{2}P_{XX}\delta^{ij}\varphi_1'\partial_iB_1\partial_j\varphi_2{\varphi_0'}^2 - \\
& \frac{1}{2}P_{XX}\delta^{ij}\varphi_0'\varphi_1'\partial_i\varphi_1\partial_j\varphi_2 + \frac{1}{8}P_{XX}{\varphi_0'}^2{\varphi_2'}^2 - \frac{1}{2}P_{XX}\delta^{ij}\varphi_2'\partial_iB_1\partial_j\varphi_1{\varphi_0'}^2 - \frac{1}{4}P_{XX}\delta^{ij}\varphi_0'\varphi_2'\partial_i\varphi_1\partial_j\varphi_1 + \\
& \frac{1}{8}P_{\varphi\varphi}\varphi_2^2a^4 + \frac{1}{4}P_{X\varphi}\phi_2{\varphi_0'}^2a^2\varphi_2 + \frac{1}{4}P_{X\varphi}\delta^{ij}\partial_iB_1\partial_jB_1{\varphi_0'}^2a^2\varphi_2 - \frac{1}{2}P_{X\varphi}\phi_1^2{\varphi_0'}^2a^2\varphi_2 + \frac{1}{2}P_{X\varphi}\phi_1\varphi_0'\varphi_1'a^2\varphi_2 + \\
& \frac{1}{2}P_{X\varphi}\phi_1\varphi_0'\varphi_2'a^2\varphi_1 - \frac{1}{4}P_{X\varphi}\varphi_0'\varphi_2'a^2\varphi_2 - \frac{1}{4}P_{X\varphi}{\varphi_1'}^2a^2\varphi_2 - \frac{1}{2}P_{X\varphi}\varphi_1'\varphi_2'a^2\varphi_1 + \frac{1}{2}P_{X\varphi}\delta^{ij}\varphi_0'\partial_iB_1\partial_j\varphi_1a^2\varphi_2 + \\
& \frac{1}{2}P_{X\varphi}\delta^{ij}\varphi_0'\partial_iB_1\partial_j\varphi_2a^2\varphi_1 + \frac{1}{4}P_{X\varphi}\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1a^2\varphi_2 + \frac{1}{2}P_{X\varphi}\delta^{ij}\partial_i\varphi_1\partial_j\varphi_2a^2\varphi_1 - \frac{1}{4}P_{XXX}\varphi_2'\phi_1^2{\varphi_0'}^5a^{-2} + \\
& \frac{1}{2}P_{XXX}\phi_1\varphi_1'\varphi_2'{\varphi_0'}^4a^{-2} - \frac{1}{4}P_{XXX}\varphi_2'{\varphi_0'}^3{\varphi_1'}^2a^{-2} + \frac{1}{4}P_{\varphi\varphi\varphi}\varphi_1^2a^4\varphi_2 + \frac{1}{4}P_{XX\varphi}\phi_1^2{\varphi_0'}^4\varphi_2 - \frac{1}{2}P_{XX\varphi}\phi_1\varphi_1'{\varphi_0'}^3\varphi_2 - \\
& \frac{1}{2}P_{XX\varphi}\phi_1\varphi_2'{\varphi_0'}^3\varphi_1 + \frac{1}{4}P_{XX\varphi}{\varphi_0'}^2{\varphi_1'}^2\varphi_2 + \frac{1}{2}P_{XX\varphi}\varphi_1'\varphi_2'{\varphi_0'}^2\varphi_1 + \frac{1}{2}P_{X\varphi\varphi}\phi_1{\varphi_0'}^2a^2\varphi_1\varphi_2 - \frac{1}{2}P_{X\varphi\varphi}\varphi_0'\varphi_1'a^2\varphi_1\varphi_2 - \\
& \frac{1}{4}P_{X\varphi\varphi}\varphi_0'\varphi_2'\varphi_1^2a^2 + \frac{1}{2}P_X\phi_1\delta^{ij}\partial_i\varphi_1\partial_j\varphi_2a^2 + \frac{1}{2}P_{\varphi\varphi}\phi_1a^4\varphi_1\varphi_2 + \frac{1}{4}P_\varphi\phi_2a^4\varphi_2 + \frac{1}{4}P_\varphi\delta^{ij}\partial_iB_1\partial_jB_1a^4\varphi_2 \\
& - \frac{1}{4}P_\varphi\phi_1^2a^4\varphi_2\Big)
\end{aligned}
\tag{2.30}
$$

Varying the action with respect to φ_2 leads to the same equation of motion of φ_2 (2.26). Similarly second order equations of ϕ_2 and B_2 can be obtained from varying fourth order action with respect to ϕ_2 and B_2 (2.24) and (2.25), respectively. This mechanism can be extended up to any order and we can generalize that, *equations obtained from both the approaches are identical and there are no ambiguities as discussed in Ref.* [50].

Another way of seeing constraints is, action (2.29) or (2.30) contain no time derivative of ϕ_1, B_1, ϕ_2 and B_2, i.e., Lapse function and Shift vector algebraically enter in the action. Hence, variation with respect to ϕ and B always lead to constraint equations. So, we can use (2.19) and (2.20) constraint equations to eliminate ϕ_1 and B_1 from the action and use background equations (2.16) and (2.17) to obtain a second order single variable action in terms of φ_1. Further, writing the action in terms of Mukhanov-Sasaki variable 'v',

$$
{}^{(2)}\mathcal{S}^{NC} = \frac{1}{2}\int d^4x\Big\{v'^2 - c_s^2\ \delta^{ij}\ \partial_iv\partial_jv + \frac{z''}{z}v^2\Big\} \tag{2.31}
$$

where v, z and c_s are defined in equation (2.23). We can vary the action (2.31) with respect to v to obtain equation of motion of v, which is identical to the equation (2.22). Hence *at first order, order-by-order Einstein's equation approach and reduced action approach lead to identical result.*

Similar procedure may be followed to obtain reduced second order single variable equation of motion. Fourth order action does not contain terms that have time derivatives of ϕ_2 and B_2. Hence, as in the first order, one should be able to substitute ϕ_2 and B_2 in the fourth order action to obtain a reduced action in terms of φ_2. Malik et al showed, for canonical scalar field under slow roll approximation, that the single variable equation from both approaches are same[54]. Similarly, since in the case of non-canonical scalar field, equations of Lapse function ϕ_2 and Shift vector ∂_iB_2 are (1) identical for both the approaches and (2)

are constraint equations, reduced single variable form of the equation of motion should also be identical. In Appendix B, we give the reduced single variable action as well as equation of motion in terms of 'φ_2' for non-canonical scalar field in Power law limit. Hence, *both approaches give the identical results up to second order.*

At the first order, equations of motion are linear in first order variables. In higher order, only the highest order perturbed variables appear linearly, where the lowest order perturbed variables contribute non-linearly to equations of motion. For example, in second order, equations are linear in second order variables φ_2, ϕ_2 and B_2 but are quadratic in first order variables φ_1, ϕ_1 and B_1. Hence, as pointed out in [50], it does appear that obtaining equations of field variables ϕ, B and φ at higher order from the two approaches may not be identical and thus the reduced form of the single variable equations of motion of the two different approaches may differ. However, *instead of the non-linear form of the perturbed action, reduced single variable equations of motion at second order obtained from both the approaches are identical, hence we can generalize that at every order, in the case of non-canonical scalar field, both approaches lead to identical result.* This leads to the following question:

Why Appignani et al[50] obtained different equations of motion from two approaches?

In the simplified model proposed in Ref.[50], authors have assumed that the homogeneous universe is filled with matter fluctuations with no Lapse function ϕ and Shift vector $\partial_i B$. They have shown that in this simplified model stress tensor, energy density and pressure are not identical for both the approaches. Note that, since there are no metric fluctuations, left hand sides of the perturbed Einstein's equations are zero. This leads to five equations ($T^0_0 = 0, T^0_i = 0$ and equation of motion of scalar field) for a single perturbed variable $\delta\varphi$, which lead to the inconsistency of the simplified model, hence the ambiguities. Another way of looking into this is the following: in the action approach, one can obtain the Hamiltonian (momentum) constraint of the system by varying the action with respect to ϕ (B). Since in this simplified model, both are not present, this leads to inconsistent results.

This leads to another important question which we address in the rest of the paper is:

For what theories of gravity and matter field, the two approaches lead to identical single variable equation of motion?

To answer the question, let us look at the procedure of conventional gauge invariant cosmological perturbation theory, which is based on two things, first, to obtain gauge invariant variables and second, to obtain a single variable action/equation of motion in terms of gauge invariant variables. Gauge invariant variables are model independent (if the background metric is unchanged), i.e., these are same for canonical, non-canonical or Galilean models so that we can always remove two variables out of five by using gauge conditions and define suitable gauge invariant variables. At each order, we start from five perturbed variables (ϕ, B, ψ, E and φ). The gauge choice helps to remove two variables. Carefully choosing a gauge (in our case, $E = 0$ and $\psi = 0$) at any order could fix the gauge issue and reduced variables will coincide with gauge invariant variables[53]. So all equations in terms of those variables also become gauge invariant.

Obtaining a single variable action/equation of motion depends solely on gauge fixing (the procedure discussed in the above paragraph) and two constraint equations which differ from model to model. If Lapse function N (ϕ in perturbed case) and Shift vector N_i ($\partial_i B$ in perturbed case) remain constraints for any models, i.e., those functions algebraically enter

into the action then equations of motion of Lapse function and Shift vector contain no time derivatives of them, we can always eliminate them from action/equation of motion to get a single variable action/equation. This helps to reduce the degrees of freedom to one and we can write the action/equation of motion in a single variable form. However, if ϕ_1 or B_1 or both become dynamical i.e., if the action contains terms containing time derivatives of Lapse function and/or Shift vector such that equations contain double time derivatives of those variables then it is not possible to substitute those variables in the action or in the equation of motion of the scalar field and the method fails. We refer the constrained nature of Lapse function and Shift vector as 'Constraint consistency' condition. If it is satisfied then the whole method of gauge invariant cosmological perturbation theory will work. In the next section, we test the 'constraint consistency' condition for several models that are used in the literature.

In fact, the whole exercise may be done in terms of Lapse N, Shift N_i and scalar field φ without applying any perturbation theory. From Hamiltonian theory of General relativity or from Einstein's equation we obtain constraint equations, i.e., Hamiltonian and Momentum constraints which are functions of $(N, N_i, \gamma_{ij}, \varphi, \varphi', \partial_i\varphi)$ in which Lapse function and Shift vector are constraints. If out of four constraint equations (1 Hamiltonian equation or 0-0 Einstein's equation and 3 Momentum constraint equation or 0-i Einstein's equation), we can solve and extract four quantity, one Lapse function and three component Shift vector then we can substitute those back in the action or in equation of motion of the scalar field. Unfortunately, GR equations are so highly non-linear that analytically solving constraint equations for Lapse function and Shift vector and obtaining a single variable action or equation is very difficult. Perturbation theory helps to simplify those equations so that we can invert those equations in terms of Lapse function and Shift vector and obtain a simplified solutions of constraint functions.

3 Specific models

In this section, first we start with well known models of inflation within the framework of general relativity and then move to modified gravity models. To check for constraint consistency we follow action formulation, write down second order action in terms of perturbed variables and identify terms that contain time derivatives of Lapse function and/or Shift vector.

3.1 Minimally coupled Galilean field

We start with the model with derivatives of metric in the action,

$$\mathcal{S}_m^G = \int d^4x\sqrt{-g}\; G(X, \varphi)\Box\varphi \tag{3.1}$$

which has been proposed by Kobayashi et al[44, 45] where '$\Box$' is defined by equation (2.2). Since '$\Box$' contains time derivatives of metric as well as matter, it is not obvious whether the action can be expressed in a single variable form. After partial integration, the second order matter action becomes as follows:

$$
\begin{aligned}
{}^{(2)}\mathcal{S}_m^G = \int d^4x \Big\{ & 5\boldsymbol{G_X\phi_1\phi_1'\varphi_0'^{3}} + \frac{15}{2}G_X\varphi_0''\phi_1^2{\varphi_0'}^2 - \frac{15}{2}G_X a'\phi_1^2{\varphi_0'}^3 a^{-1} - \boldsymbol{3G_X\phi_1'\varphi_1'\varphi_0'^{2}} - 6G_X\phi_1\varphi_0'\varphi_1'\varphi_0'' + \\
& 9G_X\phi_1\varphi_1' a'{\varphi_0'}^2 a^{-1} - G_X\delta^{ij}\partial_i B_1\partial_{0j}B_1{\varphi_0'}^3 - \frac{3}{2}G_X\delta^{ij}\partial_i B_1\partial_j B_1\varphi_0''{\varphi_0'}^2 + \frac{3}{2}G_X\delta^{ij}a'\partial_i B_1\partial_j B_1{\varphi_0'}^3 a^{-1} - \\
& 3G_X\phi_1\varphi_1''{\varphi_0'}^2 + G_X\varphi_0''{\varphi_1'}^2 + 2G_X\varphi_0'\varphi_1'\varphi_1'' - 3G_X\varphi_0' a'{\varphi_1'}^2 a^{-1} - 2G_X\delta^{ij}\varphi_0'\partial_i B_1\partial_j\varphi_1\varphi_0'' - \\
& G_X\delta^{ij}\partial_i\varphi_1\partial_{0j}B_1{\varphi_0'}^2 - 2G_X\delta^{ij}\partial_i B_1\partial_{0j}\varphi_1{\varphi_0'}^2 + 3G_X\delta^{ij}a'\partial_i B_1\partial_j\varphi_1{\varphi_0'}^2 a^{-1} - 2G_X\delta^{ij}\varphi_0'\partial_i\varphi_1\partial_{0j}\varphi_1 + \\
& G_X\delta^{ij}\varphi_0' a'\partial_i\varphi_1\partial_j\varphi_1 a^{-1} + G_X\delta^{ij}\partial_i B_1\partial_j\phi_1{\varphi_0'}^3 + G_X\delta^{ij}\partial_i\phi_1\partial_j\varphi_1{\varphi_0'}^2 - \boldsymbol{G_{XX}\phi_1\phi_1'\varphi_0'^{5}a^{(-2)}} - \\
& 5G_{XX}\varphi_0''\phi_1^2{\varphi_0'}^4 a^{(-2)} + 5G_{XX}a'\phi_1^2{\varphi_0'}^5 a^{(-3)} + 7G_{XX}\phi_1\varphi_1'\varphi_0''{\varphi_0'}^3 a^{(-2)} - 8G_{XX}\phi_1\varphi_1' a'{\varphi_0'}^4 a^{(-3)} + \\
& G_{XX}\phi_1\varphi_1''{\varphi_0'}^4 a^{(-2)} + \frac{1}{2}G_{XX}\delta^{ij}\partial_i B_1\partial_j B_1\varphi_0''{\varphi_0'}^4 a^{(-2)} - \frac{1}{2}G_{XX}\delta^{ij}a'\partial_i B_1\partial_j B_1{\varphi_0'}^5 a^{(-3)} + \boldsymbol{G_{XX}\phi_1'\varphi_1'\varphi_0'^{4}a^{(-2)}} \\
& - \frac{5}{2}G_{XX}\varphi_0''{\varphi_0'}^2{\varphi_1'}^2 a^{(-2)} + \frac{7}{2}G_{XX}a'{\varphi_0'}^3{\varphi_1'}^2 a^{(-3)} - G_{XX}\varphi_1'\varphi_1''{\varphi_0'}^3 a^{(-2)} + G_{XX}\delta^{ij}\partial_i B_1\partial_j\varphi_1\varphi_0''{\varphi_0'}^3 a^{(-2)} - \\
& G_{XX}\delta^{ij}a'\partial_i B_1\partial_j\varphi_1{\varphi_0'}^4 a^{(-3)} + \frac{1}{2}G_{XX}\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1\varphi_0''{\varphi_0'}^2 a^{(-2)} - \frac{1}{2}G_{XX}\delta^{ij}a'\partial_i\varphi_1\partial_j\varphi_1{\varphi_0'}^3 a^{(-3)} - \\
& \boldsymbol{G_{XY}\phi_1'\varphi_0'^{3}\varphi_1} - 3G_{XY}\phi_1\varphi_0''{\varphi_0'}^2\varphi_1 + 3G_{XY}\phi_1 a'{\varphi_0'}^3 a^{-1}\varphi_1 + 2G_{XY}\varphi_0'\varphi_1'\varphi_0''\varphi_1 - 3G_{XY}\varphi_1' a'{\varphi_0'}^2 a^{-1}\varphi_1 + \\
& G_{XY}\varphi_1''{\varphi_0'}^2\varphi_1 + \frac{1}{2}G_{XXX}\varphi_0''\phi_1^2{\varphi_0'}^6 a^{(-4)} - \frac{1}{2}G_{XXX}a'\phi_1^2{\varphi_0'}^7 a^{(-5)} - G_{XXX}\phi_1\varphi_1'\varphi_0''{\varphi_0'}^5 a^{(-4)} + \\
& G_{XXX}\phi_1\varphi_1' a'{\varphi_0'}^6 a^{(-5)} + \frac{1}{2}G_{XXX}\varphi_0''{\varphi_0'}^4{\varphi_1'}^2 a^{(-4)} - \frac{1}{2}G_{XXX}a'{\varphi_0'}^5{\varphi_1'}^2 a^{(-5)} + \frac{1}{2}G_{XYY}\varphi_0''{\varphi_0'}^2\varphi_1^2 - \\
& \frac{1}{2}G_{XYY}a'{\varphi_0'}^3\varphi_1^2 a^{-1} + G_{XXY}\phi_1\varphi_0''{\varphi_0'}^4 a^{(-2)}\varphi_1 - G_{XXY}\phi_1 a'{\varphi_0'}^5 a^{(-3)}\varphi_1 - G_{XXY}\varphi_1'\varphi_0''{\varphi_0'}^3 a^{(-2)}\varphi_1 + \\
& G_{XXY}\varphi_1' a'{\varphi_0'}^4 a^{(-3)}\varphi_1 + \frac{1}{4}G_Y\delta^{ij}\partial_i B_1\partial_j B_1{\varphi_0'}^2 a^2 - \frac{3}{4}G_Y\phi_1^2{\varphi_0'}^2 a^2 + G_Y\phi_1\varphi_0'\varphi_1' a^2 - \frac{1}{2}G_Y{\varphi_1'}^2 a^2 + \\
& G_Y\delta^{ij}\varphi_0'\partial_i B_1\partial_j\varphi_1 a^2 + \frac{1}{2}G_Y\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1 a^2 + \frac{3}{2}G_{XY}\phi_1^2{\varphi_0'}^4 - \frac{5}{2}G_{XY}\phi_1\varphi_1'{\varphi_0'}^3 - \frac{1}{4}G_{XY}\delta^{ij}\partial_i B_1\partial_j B_1{\varphi_0'}^4 + \\
& \frac{5}{4}G_{XY}{\varphi_0'}^2{\varphi_1'}^2 - \frac{1}{2}G_{XY}\delta^{ij}\partial_i B_1\partial_j\varphi_1{\varphi_0'}^3 - \frac{1}{4}G_{XY}\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1{\varphi_0'}^2 + \frac{1}{2}G_{YY}\phi_1{\varphi_0'}^2 a^2\varphi_1 - G_{YY}\varphi_0'\varphi_1' a^2\varphi_1 \\
& - \frac{1}{4}G_{XXY}\phi_1^2{\varphi_0'}^6 a^{(-2)} + \frac{1}{2}G_{XXY}\phi_1\varphi_1'{\varphi_0'}^5 a^{(-2)} - \frac{1}{4}G_{XXY}{\varphi_0'}^4{\varphi_1'}^2 a^{(-2)} - \frac{1}{4}G_{YYY}{\varphi_0'}^2\varphi_1^2 a^2 - \\
& \frac{1}{2}G_{XYY}\phi_1{\varphi_0'}^4\varphi_1 + \frac{1}{2}G_{XYY}\varphi_1'{\varphi_0'}^3\varphi_1 a^2 \Big\}
\end{aligned}
\tag{3.2}
$$

where $G_X \equiv \partial_X G, G_Y \equiv \partial_\varphi G$ for background and so on. The derivatives of the constraints (ϕ_1) appear linearly in the above reduced action (those terms are highlighted in the above expression). However, by performing partial integration one can rewrite these terms as terms proportional to ϕ_1 e.g., first terms in the action can be written as $-\frac{5}{2}\partial_0\{G_X{\varphi_0'}^3\}\phi_1^2$. So, *although the action contains time derivative of Lapse function but it is reducible to action with no time derivative of Lapse or Shift, hence, variation of these terms do not lead constraint inconsistencies.*

3.2 $(\varphi_{;\lambda}\varphi^{;\lambda}\{\Box\varphi\}^2)$ model

Let us consider the following model where the matter action is given by

$$
\mathcal{S}_m = \int d^4x\sqrt{-g}\varphi_{;\lambda}\varphi^{;\lambda}\{\Box\varphi\}^2 \tag{3.3}
$$

and is minimally coupled to gravity[1]. Expanding the matter action to second order, we get,

[1]Note that, our motivation is only about the consistency of the method discussed in the first section for different models, not to check its physical observational viability or any other problems such as Higher order Ostrogradsky's ghost

$$
\begin{aligned}
{}^{(2)}\mathcal{S}_m = \int d^4x\{ & -\frac{35}{2}\phi_1^2\varphi_0'^2\varphi_0''^2a^{-2} - 70a'\varphi_0''\phi_1^2\varphi_0'^3a^{-3} - 14\phi_1\phi_1'\varphi_0''\varphi_0'^3a^{-2} - 10\phi_1\delta^{ij}\varphi_0''\partial_{ij}B_1\varphi_0'^3a^{-2} - \\
& 70\phi_1^2\varphi_0'^4a'^2a^{-4} - 28\phi_1\phi_1'a'\varphi_0'^4a^{-3} - 20\phi_1\delta^{ij}a'\partial_{ij}B_1\varphi_0'^4a^{-3} + 10\phi_1\varphi_0''\varphi_1''\varphi_0'^2a^{-2} - 6\phi_1\delta^{ij}\varphi_0''\partial_{ji}\varphi_1\varphi_0'^2a^{-2} \\
& + 20\phi_1a'\varphi_1''\varphi_0'^3a^{-3} + 60\phi_1\varphi_1'a'\varphi_0''\varphi_0'^2a^{-3} - 12\phi_1\delta^{ij}a'\partial_{ji}\varphi_1\varphi_0'^3a^{-3} + 80\phi_1\varphi_1'\varphi_0'^3a'^2a^{-4} + 10\phi_1\varphi_0'\varphi_1'\varphi_0''^2a^{-2} \\
& + 6\phi_1'\varphi_1'\varphi_0''\varphi_0'^2a^{-2} + 6\delta^{ij}\varphi_1'\varphi_0''\partial_{ij}B_1\varphi_0'^2a^{-2} + 16\phi_1'\varphi_1'a'\varphi_0'^3a^{-3} + 16\delta^{ij}\varphi_1'a'\partial_{ij}B_1\varphi_0'^3a^{-3} - 4\varphi_0'\varphi_1'\varphi_0''\varphi_1''a^{-2} \\
& + 4\delta^{ij}\varphi_0'\varphi_1'\varphi_0''\partial_{ji}\varphi_1a^{-2} - 12\varphi_1'a'\varphi_1''\varphi_0'^2a^{-3} - 12\varphi_0'a'\varphi_0''\varphi_1'^2a^{-3} + 12\delta^{ij}\varphi_1'a'\partial_{ji}\varphi_1\varphi_0'^2a^{-3} - 24\varphi_0'^2\varphi_1'^2a'^2a^{-4} \\
& + \frac{5}{2}\delta^{ij}\partial_iB_1\partial_jB_1\varphi_0'^2\varphi_0''^2a^{-2} + 10\delta^{ij}a'\partial_iB_1\partial_jB_1\varphi_0''\varphi_0'^3a^{-3} + 10\delta^{ij}\partial_iB_1\partial_jB_1\varphi_0'^4a'^2a^{-4} - \varphi_1'^2\varphi_0''^2a^{-2} + \\
& 2\delta^{ij}\varphi_0'\partial_jB_1\partial_i\varphi_1\varphi_0''^2a^{-2} + 8\delta^{ij}a'\partial_jB_1\partial_i\varphi_1\varphi_0''\varphi_0'^2a^{-3} + 8\delta^{ij}\partial_jB_1\partial_i\varphi_1\varphi_0'^3a'^2a^{-4} + \delta^{ij}\partial_i\varphi_1\partial_j\varphi_1\varphi_0''^2a^{-2} + \\
& 4\delta^{ij}\varphi_0'a'\partial_i\varphi_1\partial_j\varphi_1\varphi_0''a^{-3} + 4\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1\varphi_0'^2a'^2a^{-4} + 4\delta^{ij}\partial_iB_1\varphi_0''\partial_{j0}\varphi_1\varphi_0'^2a^{-2} + 2\phi_1'\varphi_1''\varphi_0'^3a^{-2} + \\
& 4\delta^{ij}a'\partial_iB_1\partial_j\varphi_1\varphi_0''\varphi_0'^2a^{-3} + 2\delta^{ij}\partial_i\varphi_1\varphi_0''\partial_{0j}B_1\varphi_0'^2a^{-2} + 2\delta^{ij}\varphi_1''\partial_{ij}B_1\varphi_0'^3a^{-2} + 8\delta^{ij}a'\partial_iB_1\partial_{j0}\varphi_1\varphi_0'^3a^{-3} - \\
& 2\delta^{ij}\phi_1'\partial_{ji}\varphi_1\varphi_0'^3a^{-2} - 2\delta^{ij}\delta^{kl}\partial_{ij}B_1\partial_{lk}\varphi_1\varphi_0'^3a^{-2} + 2\delta^{ij}\partial_i\phi_1\partial_j\varphi_1\varphi_0''\varphi_0'^2a^{-2} + 8\delta^{ij}\partial_iB_1\partial_j\varphi_1\varphi_0'^3a'^2a^{-4} + \\
& 4\delta^{ij}a'\partial_i\varphi_1\partial_{0j}B_1\varphi_0'^3a^{-3} + 4\delta^{ij}a'\partial_i\phi_1\partial_j\varphi_1\varphi_0'^3a^{-3} + 2\delta^{ij}\partial_iB_1\varphi_0''\partial_{0j}B_1\varphi_0'^3a^{-2} - 2\delta^{ij}\partial_iB_1\partial_j\phi_1\varphi_0''\varphi_0'^3a^{-2} - \\
& \boldsymbol{\phi_1'^2\varphi_0'^4a^{-2}} + 4\delta^{ij}a'\partial_iB_1\partial_{0j}B_1\varphi_0'^4a^{-3} - 4\delta^{ij}a'\partial_iB_1\partial_j\phi_1\varphi_0'^4a^{-3} - 2\delta^{ij}\phi_1'\partial_{ij}B_1\varphi_0'^4a^{-2} - \\
& \delta^{ij}\delta^{kl}\partial_{ij}B_1\partial_{kl}B_1\varphi_0'^4a^{-2} - \varphi_0'^2\varphi_1''^2a^{-2} + 2\delta^{ij}\varphi_1''\partial_{ji}\varphi_1\varphi_0'^2a^{-2} - \delta^{ij}\delta^{kl}\partial_{ji}\varphi_1\partial_{lk}\varphi_1\varphi_0'^2a^{-2}\}
\end{aligned}
\tag{3.4}
$$

Following points are worth-noting regarding the above action: (i) Unlike Galilean scalar fields, the above action contains square of the derivative of constraints (ϕ'^2) [the term is highlighted in the above expression], (ii) In case of Galilean, it was possible to rewrite it as boundary term, in this case it is not possible. This implies that using reduced action approach, we cannot obtain a single dynamical equation. Hence as discussed in section (2), the constraint consistency is not satisfied.

3.3 $(\varphi_{;\lambda}\varphi^{;\lambda}\ \varphi_{;\mu\nu}\varphi^{;\mu\nu})$ model

As like the previous subsection, the following action also contains higher time derivatives of constraints. Expanding the matter action

$$
\mathcal{S}_m = \int d^4x\ \sqrt{-g}\ \varphi_{;\lambda}\varphi^{;\lambda}\ \varphi_{;\mu\nu}\varphi^{;\mu\nu} \tag{3.5}
$$

to second order, we get,

$$
\begin{aligned}
{}^{(2)}\mathcal{S}_m = \int d^4x\{ & -\frac{35}{2}\phi_1^2\varphi_0'^2\varphi_0''^2a^{-2} + 35a'\varphi_0''\phi_1^2\varphi_0'^3a^{-3} - 14\phi_1\phi_1'\varphi_0''\varphi_0'^3a^{-2} - 70\phi_1^2\varphi_0'^4a'^2a^{-4} + 14\phi_1\phi_1'a'\varphi_0'^4a^{-3} - \\
& 10\phi_1\delta^{ij}a'\partial_{ij}B_1\varphi_0'^4a^{-3} + 10\phi_1\varphi_0''\varphi_1''\varphi_0'^2a^{-2} - 10\phi_1a'\varphi_1''\varphi_0'^3a^{-3} - 30\phi_1\varphi_1'a'\varphi_0''\varphi_0'^2a^{-3} - 6\phi_1\delta^{ij}a'\partial_{ji}\varphi_1\varphi_0'^3a^{-3} \\
& + 80\phi_1\varphi_1'\varphi_0'^3a'^2a^{-4} + 10\phi_1\varphi_0'\varphi_1'\varphi_0''^2a^{-2} + 6\phi_1'\varphi_1'\varphi_0''\varphi_0'^2a^{-2} - 8\phi_1'\varphi_1'a'\varphi_0'^3a^{-3} + 8\delta^{ij}\varphi_1'a'\partial_{ij}B_1\varphi_0'^3a^{-3} - \\
& 4\varphi_0'\varphi_1'\varphi_0''\varphi_1''a^{-2} + 6\varphi_1'a'\varphi_1''\varphi_0'^2a^{-3} + 6\varphi_0'a'\varphi_0''\varphi_1'^2a^{-3} + 6\delta^{ij}\varphi_1'a'\partial_{ji}\varphi_1\varphi_0'^2a^{-3} - 24\varphi_0'^2\varphi_1'^2a'^2a^{-4} + \\
& \frac{5}{2}\delta^{ij}\partial_iB_1\partial_jB_1\varphi_0'^2\varphi_0''^2a^{-2} - 5\delta^{ij}a'\partial_iB_1\partial_jB_1\varphi_0''\varphi_0'^3a^{-3} + 10\delta^{ij}\partial_iB_1\partial_jB_1\varphi_0'^4a'^2a^{-4} - \varphi_1'^2\varphi_0''^2a^{-2} + \\
& 2\delta^{ij}\varphi_0'\partial_jB_1\partial_i\varphi_1\varphi_0''^2a^{-2} - 4\delta^{ij}a'\partial_jB_1\partial_i\varphi_1\varphi_0''\varphi_0'^2a^{-3} + 8\delta^{ij}\partial_jB_1\partial_i\varphi_1\varphi_0'^3a'^2a^{-4} + \delta^{ij}\partial_i\varphi_1\partial_j\varphi_1\varphi_0''^2a^{-2} - \\
& 2\delta^{ij}\varphi_0'a'\partial_i\varphi_1\partial_j\varphi_1\varphi_0''a^{-3} + 6\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1\varphi_0'^2a'^2a^{-4} + 4\delta^{ij}\partial_iB_1\varphi_0''\partial_{j0}\varphi_1\varphi_0'^2a^{-2} + 2\phi_1'\varphi_1''\varphi_0'^3a^{-2} - \\
& 2\delta^{ij}a'\partial_iB_1\partial_j\varphi_1\varphi_0''\varphi_0'^2a^{-3} + 2\delta^{ij}\partial_i\varphi_1\varphi_0''\partial_{0j}B_1\varphi_0'^2a^{-2} - 4\delta^{ij}a'\partial_iB_1\partial_{j0}\varphi_1\varphi_0'^3a^{-3} - 4\delta^{ij}\partial_i\phi_1\partial_{j0}\varphi_1\varphi_0'^3a^{-2} - \\
& 2\delta^{ij}\delta^{kl}\partial_{ik}B_1\partial_{lj}\varphi_1\varphi_0'^3a^{-2} + 2\delta^{ij}\partial_i\phi_1\partial_j\varphi_1\varphi_0''\varphi_0'^2a^{-2} + 8\delta^{ij}\partial_iB_1\partial_j\varphi_1\varphi_0'^3a'^2a^{-4} - 2\delta^{ij}a'\partial_i\varphi_1\partial_{0j}B_1\varphi_0'^3a^{-3} + \\
& 2\delta^{ij}a'\partial_i\phi_1\partial_j\varphi_1\varphi_0'^3a^{-3} + 2\delta^{ij}\partial_iB_1\varphi_0''\partial_{0j}B_1\varphi_0'^3a^{-2} - 2\delta^{ij}\partial_iB_1\partial_j\phi_1\varphi_0''\varphi_0'^3a^{-2} - \boldsymbol{\phi_1'^2\varphi_0'^4a^{-2}} - \\
& 2\delta^{ij}a'\partial_iB_1\partial_{0j}B_1\varphi_0'^4a^{-3} + 2\delta^{ij}a'\partial_iB_1\partial_j\phi_1\varphi_0'^4a^{-3} + 2\delta^{ij}\partial_i\phi_1\partial_j\phi_1\varphi_0'^4a^{-2} - \delta^{ij}\delta^{kl}\partial_{ik}B_1\partial_{jl}B_1\varphi_0'^4a^{-2} - \\
& \varphi_0'^2\varphi_1''^2a^{-2} + 2\delta^{ij}\partial_{i0}\varphi_1\partial_{j0}\varphi_1\varphi_0'^2a^{-2} - \delta^{ij}\delta^{kl}\partial_{ki}\varphi_1\partial_{lj}\varphi_1\varphi_0'^2a^{-2} - 4\delta^{ij}a'\partial_i\varphi_1\partial_{j0}\varphi_1\varphi_0'^2a^{-3}\}
\end{aligned}
\tag{3.6}
$$

It is important to note that the second order action contains ${\phi'}^2$ which cannot be absorbed as a boundary term (the highlighted term in the above expression). Hence the constraint condition is not satisfied leading to the fact that the reduced action does not lead to single variable equation of motion.

From the above analysis, we can generalize and apply this to any higher derivative scalar theory models like $\{\Box\varphi\}^3$, $\Box\Box\varphi$, $\varphi^{;\alpha}_{;\beta}\varphi^{;\beta}_{;\gamma}\varphi^{;\gamma}_{;\alpha}$, $\varphi_{;\alpha\beta\gamma\delta}\varphi^{\alpha\beta\gamma\delta}$ and that obtaining a single variable equation of motion or action is not possible for these kind of models.

3.4 $f(R)$ model

Until now, we have considered different forms of scalar field action without modifying gravity. In this subsection, we consider the simplest modification i.e. $R + \alpha R^2$ while we consider the matter to be canonical scalar field. Since R and matter part of the action does not have any inconsistency, we expand R^2 up to second order in perturbed variables to get,

$$\begin{aligned}
&36\delta^{ij}\delta^{kl}\partial_{ij}B_1\partial_{kl}B_1{a'}^2a^{-6} + 12\delta^{ij}\delta^{kl}a'\partial_{ij}B_1\partial_{0kl}B_1a^{-5} + 72\delta^{ij}\phi_1'\partial_{ij}B_1{a'}^2a^{-6}+\\
&288\phi_1\delta^{ij}a'a''\partial_{ij}B_1a^{-6} + 12\delta^{ij}\delta^{kl}a'\partial_{ij}B_1\partial_{kl}\phi_1a^{-5} + 12\delta^{ij}\delta^{kl}a'\partial_{kl}B_1\partial_{0ij}B_1a^{-5} + 4\delta^{ij}\delta^{kl}\partial_{0ij}B_1\partial_{0kl}B_1a^{-4}+\\
&24\delta^{ij}\phi_1'a'\partial_{0ij}B_1a^{-5} + 96\phi_1\delta^{ij}a''\partial_{0ij}B_1a^{-5} + 4\delta^{ij}\delta^{kl}\partial_{kl}\phi_1\partial_{0ij}B_1a^{-4} + \boldsymbol{36{\phi_1'}^2{a'}^2a^{-6}}+\\
&432\phi_1\phi_1'a'a''a^{-6} + 24\delta^{ij}\phi_1'a'\partial_{ij}\phi_1a^{-5} + 432\phi_1^2{a''}^2a^{-6} + 96\phi_1\delta^{ij}a''\partial_{ij}\phi_1a^{-5}+\\
&12\delta^{ij}\delta^{kl}a'\partial_{kl}B_1\partial_{ij}\phi_1a^{-5} + 4\delta^{ij}\delta^{kl}\partial_{ij}\phi_1\partial_{0kl}B_1a^{-4} + 4\delta^{ij}\delta^{kl}\partial_{ij}\phi_1\partial_{kl}\phi_1a^{-4} + 24\delta^{ij}\phi_1'a''\partial_{ij}B_1a^{-5}-\\
&72\delta^{ij}a'\partial_iB_1a''\partial_{0j}B_1a^{-6} + 12\delta^{ij}\delta^{kl}a''\partial_{ij}B_1\partial_{kl}B_1a^{-5} + 72\delta^{ij}a'\partial_iB_1\partial_j\phi_1a''a^{-6} - 72\delta^{ij}\partial_iB_1\partial_jB_1{a''}^2a^{-6}+\\
&24\delta^{ij}\partial_i\phi_1\partial_j\phi_1a''a^{-5} - 12\delta^{ij}\delta^{kl}a''\partial_{ik}B_1\partial_{jl}B_1a^{-5}
\end{aligned} \tag{3.7}$$

Following points are interesting to note from the above expression: (i) R^2 term contains time derivative of ϕ, that cannot be absorbed as a boundary term. This implies that the above action cannot lead to the constraint equation. (ii) By doing a conformal transformation $g_{\mu\nu} \to \tilde{g_{\mu\nu}} = \Omega^2 g_{\mu\nu}$, the term containing the time derivative of ϕ can be absorbed as a matter field and hence the constraint equation recovered. (iii) The constraint consistency allows us to identify that the $f(R)$ gravity models, without conformal transformation lead to inconsistent dynamics.

3.5 $[\varphi_{;\lambda}\varphi^{;\lambda}(\{\Box\varphi\}^2 - \varphi_{;\mu\nu}\varphi^{;\mu\nu})]$ model

As we have shown above, certain higher derivative models do not satisfy 'constraint consistency'. We have also shown which terms in the second order action spoil the 'constraint consistency'. However, it is interesting to note the terms that contain time derivative of Lapse functions are identical. Let us consider the following scalar field action

$$S = \int d^4x\sqrt{-g}\ [\ \varphi_{;\lambda}\varphi^{;\lambda}\ \left(\{\Box\varphi\}^2 - \varphi_{;\mu\nu}\varphi^{;\mu\nu}\right)\ ] \tag{3.8}$$

The second order action is given by,

$$
\begin{aligned}
{}^{(2)}\mathcal{S}_m = \int d^4x\{ & -105a'\varphi_0''\phi_1^2{\varphi_0'}^3a^{-3} - 10\phi_1\delta^{ij}\varphi_0''\partial_{ij}B_1{\varphi_0'}^3a^{-2} - 42\phi_1\phi_1'a'{\varphi_0'}^4a^{-3} - 10\phi_1\delta^{ij}a'\partial_{ij}B_1{\varphi_0'}^4a^{-3} - \\
& 6\phi_1\delta^{ij}\varphi_0''\partial_{ji}\varphi_1{\varphi_0'}^2a^{-2} + 30\phi_1a'\varphi_1''{\varphi_0'}^3a^{-3} + 90\phi_1\varphi_1'a'\varphi_0''{\varphi_0'}^2a^{-3} - 6\phi_1\delta^{ij}a'\partial_{ji}\varphi_1{\varphi_0'}^3a^{-3} + \\
& 6\delta^{ij}\varphi_1'\varphi_0''\partial_{ij}B_1{\varphi_0'}^2a^{-2} + 24\phi_1'\varphi_1'a'{\varphi_0'}^3a^{-3} + 8\delta^{ij}\varphi_1'a'\partial_{ij}B_1{\varphi_0'}^3a^{-3} + 4\delta^{ij}\varphi_0'\varphi_1'\varphi_0''\partial_{ji}\varphi_1a^{-2} - \\
& 18\varphi_1'a'\varphi_1''{\varphi_0'}^2a^{-3} - 18\varphi_0'a'\varphi_0''{\varphi_1'}^2a^{-3} + 6\delta^{ij}\varphi_1'a'\partial_{ji}\varphi_1{\varphi_0'}^2a^{-3} + 15\delta^{ij}a'\partial_iB_1\partial_jB_1\varphi_0''{\varphi_0'}^3a^{-3} + \\
& 12\delta^{ij}a'\partial_jB_1\partial_i\varphi_1\varphi_0''{\varphi_0'}^2a^{-3} + 6\delta^{ij}\varphi_0'a'\partial_i\varphi_1\partial_j\varphi_1\varphi_0''a^{-3} - 2\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1{\varphi_0'}^2{a'}^2a^{-4} + 6\delta^{ij}a'\partial_iB_1\partial_j\varphi_1\varphi_0''{\varphi_0'}^2a^{-3} + \\
& 2\delta^{ij}\varphi_1''\partial_{ij}B_1{\varphi_0'}^3a^{-2} + 12\delta^{ij}a'\partial_iB_1\partial_{j0}\varphi_1{\varphi_0'}^3a^{-3} - 2\delta^{ij}\phi_1'\partial_{ji}\varphi_1{\varphi_0'}^3a^{-2} - 2\delta^{ij}\delta^{kl}\partial_{ij}B_1\partial_{lk}\varphi_1{\varphi_0'}^3a^{-2} + \\
& 6\delta^{ij}a'\partial_i\varphi_1\partial_{0j}B_1{\varphi_0'}^3a^{-3} + 2\delta^{ij}a'\partial_i\phi_1\partial_j\varphi_1{\varphi_0'}^3a^{-3} + 6\delta^{ij}a'\partial_iB_1\partial_{0j}B_1{\varphi_0'}^4a^{-3} - 6\delta^{ij}a'\partial_iB_1\partial_j\phi_1{\varphi_0'}^4a^{-3} - \\
& 2\delta^{ij}\phi_1'\partial_{ij}B_1{\varphi_0'}^4a^{-2} - \delta^{ij}\delta^{kl}\partial_{ij}B_1\partial_{kl}B_1{\varphi_0'}^4a^{-2} + 2\delta^{ij}\varphi_1''\partial_{ji}\varphi_1{\varphi_0'}^2a^{-2} - \delta^{ij}\delta^{kl}\partial_{ji}\varphi_1\partial_{lk}\varphi_1{\varphi_0'}^2a^{-2} + \\
& 4\delta^{ij}\partial_i\phi_1\partial_{j0}\varphi_1{\varphi_0'}^3a^{-2} + 2\delta^{ij}\delta^{kl}\partial_{ik}B_1\partial_{lj}\varphi_1{\varphi_0'}^3a^{-2} - 2\delta^{ij}\partial_i\phi_1\partial_j\phi_1{\varphi_0'}^4a^{-2} + \delta^{ij}\delta^{kl}\partial_{ik}B_1\partial_{jl}B_1{\varphi_0'}^4a^{-2} - \\
& 2\delta^{ij}\partial_{i0}\varphi_1\partial_{j0}\varphi_1{\varphi_0'}^2a^{-2} + \delta^{ij}\delta^{kl}\partial_{ki}\varphi_1\partial_{lj}\varphi_1{\varphi_0'}^2a^{-2} + 4\delta^{ij}a'\partial_i\varphi_1\partial_{j0}\varphi_1{\varphi_0'}^2a^{-3}\}
\end{aligned}
\tag{3.9}
$$

It is interesting to note the following points: (i) the action does not contain terms having time derivative of Lapse function/Shift vector. Hence the resultant equation leads to constraint equation. Similarly, $[\varphi_{;\lambda}\varphi^{;\lambda}(\{\Box\varphi\}^3 - 3\Box\varphi\ \varphi_{;\mu\nu}\varphi^{;\mu\nu} + 2\varphi^{;\alpha}_{;\beta}\varphi^{;\beta}_{;\gamma}\varphi^{;\gamma}_{;\alpha})]$ model do not have dynamical Lapse/Shift. (ii) From the above, one may be tempted to relate the constraint consistency with Ostrogradsky's instabilities. To go about checking this, let us look at the zeroth order action for the matter field, i.e.,

$$
{}^{(0)}\mathcal{S}_m = \int d^4x\{-6a'\varphi_0''{\varphi_0'}^3a^{(-3)}\} \tag{3.10}
$$

which, after integration by-parts, can be re-written as,

$$
{}^{(0)}\mathcal{S}_m = \int d^4x\{\frac{3}{2}a''a^{-3}{\varphi_0'}^4 - \frac{9}{2}{a'}^2a^{-4}{\varphi_0'}^4\} \tag{3.11}
$$

Hence the equations of motion will contain a''' and ϕ_0'''. This immediately signals Ostrogradsky's instability.

This leads us to the important conclusion: *While all 'constraint' inconsistent models have higher order Ostrogradsky's instabilities but the reverse is not true. One can have models with constraint Lapse function and Shift vector, though it may have Ostrogradsky's instabilities.*

3.6 Constraint consistent models without instabilities

In the previous subsection, we showed that identifying the terms in (3.4) and (3.6) that contains higher derivatives of Lapse function, we can remove these terms by combination of these terms two terms. However, we noticed that such actions suffer from Ostrogradsky's instabilities. The term that leads to Ostrogradsky's instability is $\frac{3}{2}a''a^{-3}{\varphi_0'}^4$ at zeroth order action (3.11). In order to cancel such term, without involving any higher derivatives of ϕ, one needs to introduce non-minimal coupling of the kinetic term of the scalar field with Ricci scalar. It is easy to show check that the term $-\frac{1}{4}(\varphi_{;\alpha}\varphi^{;\alpha})^2R$ exactly cancels this and the resulting background action becomes,

$$
{}^{(0)}\mathcal{S}_m = -\frac{9}{2}\int d^4x\ {a'}^2a^{-4}{\varphi_0'}^4 \tag{3.12}
$$

Similarly, $[\varphi_{;\lambda}\varphi^{;\lambda}(\{\Box\varphi\}^3 - 3\Box\varphi\ \varphi_{;\mu\nu}\varphi^{;\mu\nu} + 2\varphi^{;\alpha}_{;\beta}\varphi^{;\beta}_{;\gamma}\varphi^{;\gamma}_{;\alpha} - 6G^{\mu\nu}\varphi^{;\alpha}_{;\mu}\varphi_{;\alpha}\varphi_{;\nu})]$ does not have any constraint inconsistencies as well as instabilities.

The following points are worth noting regarding this:

i. We have arrived at the action $[\varphi_{;\lambda}\varphi^{;\lambda}(\{\Box\varphi\}^2 - \varphi_{;\mu\nu}\varphi^{;\mu\nu} - \frac{1}{4}(\varphi_{;\alpha}\varphi^{;\alpha})^2 R]$ by using the condition that the action does not have time derivatives of Lapse function and Shift vector and later using the additional condition that Ostrogradsky's instability does not arise.

ii. In a different manner, Nicolis et al[46] and Deffayet et al[47] have come up with the same action, only with condition of removing Ostrogradsky's instability. Here we have verified that those models will result in consistent dynamics with 'constraint consistency' and lead to single variable action as well as equation of motion.

iii. Similarly, we find that the third order derivative model is $[\varphi_{;\lambda}\varphi^{;\lambda}(\{\Box\varphi\}^3 - 3\Box\varphi\ \varphi_{;\mu\nu}\varphi^{;\mu\nu} + 2\varphi^{;\alpha}_{;\beta}\varphi^{;\beta}_{;\gamma}\varphi^{;\gamma}_{;\alpha} - 6G^{\mu\nu}\varphi^{;\alpha}_{;\mu}\varphi_{;\alpha}\varphi_{;\nu})]$ which, again has been derived in [46] and [47]. This model is also constraint consistent and free of Ostrogradsky's instability.

4 Conclusions

In this work, we have revisited the two approaches in cosmological perturbation theory — order-by-order approach of the Einstein's equation and reduced action formalism. Equivalence of the two approaches were not clear since Gravity equations are highly non-linear. In this work, we have established that equations arising from order-by-order approach of the Einstein's equations and those from the action formalism are equivalent for canonical as well as non-canonical scalar fields up to second order. These results may easily be extended to any model in Gravity models at any order.

To compare both the approaches, We have identified a 'Constraint consistency condition' where the constrained nature of Lapse function and Shift vector are studied. We have shown that, in order to obtain a reduced equation for both of the approaches, 'Constraint consistency' relation has to be satisfied, i.e., those variables should appear in the action algebraically, and no non-reducible partial time derivatives of Lapse function and Shift vector should be present. In other words, equations of motion of Lapse function and Shift vector should not contain second order partial time derivatives. We then investigated the higher order derivative theories of gravity and found that models which satisfy the constraint conditions can be applied to the conventional perturbation theory where we express all equations in a simplified form with a single variable (Curvature perturbation $\mathcal{R}$ or Mukhanov-Sasaki variable) equation of motion. One common problem with higher order derivative theories is that, they may have Higher derivative equations of motion which can give rise to Ostrogradsky's instabilities. We showed that all the models which do not satisfy 'Constraint consistency' conditions suffer form Ostrogradsky's instabilities. However, we also find that, there exist some models which satisfy constraint conditions though they have instabilities, i.e., the action can be reduced in a single variable form but the single variable equation of motion will contain higher order time derivatives. The method we have proposed here is fast and efficient and would be useful for inflationary model building.

We have also constructed some higher derivative models in such a way that those models should satisfy constraint consistency condition and can be free from higher order time derivatives. Those models have already been derived in the literature in a different manner

where the approach of constructing models are different. This ensures that, conventional perturbation theory can be used in higher derivative models which are free from Ostrogradsky's instabilities to obtain a gauge invariant single variable equation of motion using any approaches. We summarize the main results,

1. The two approaches are completely equivalent.
2. Not all models with Lagrangian density $\mathcal{L} = \int \sqrt{-g}\left\{R + F(\varphi, \partial\varphi, \partial^2\varphi)\right\}$ or $\mathcal{L} = \int \sqrt{-g}F(R, \partial\varphi, \partial^2\varphi)$ can be reduced in a single variable form.
3. To obtain a single variable form, 'Constraint consistency' condition has to be satisfied.
4. Models with constraint inconsistency show Ostrogradsky's instability but the reverse is not true.

The analysis here may have implications for models of quantum cosmology with scalar fields[55]. The quantum corrections to the matter and gravity can be modelled as effective stress-tensor [56]. The effective classical equation must also satisfy the constraint consistency. This might help to constraint the quantum cosmology models[56].

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A Coefficients of second order equation of motion of non-canonical scalar field

$$
\begin{aligned}
C_X =&\varphi_2'' - 2\phi_1\varphi_1'' - \phi_2\varphi_0'' - \nabla^2\varphi_2 - 2\phi_1'\varphi_1' - \phi_2\varphi_0' + 2\mathcal{H}\varphi_2' + 3\varphi_0''\phi_1^2 - 2\phi_1\nabla^2\varphi_1 + 6\phi_1\phi_1'\varphi_0' - \\
&4\mathcal{H}\phi_1\varphi_1' - 2\mathcal{H}\phi_2\varphi_0' - \varphi_0'\nabla^2 B_2 - 2\varphi_1'\nabla^2 B_1 - 4\delta^{ij}\partial_i B_1\partial_j\varphi_1' - 2\delta^{ij}\partial_i\phi_1\partial_j\varphi_1 - 2\delta^{ij}\partial_i\varphi_1\partial_j B_1' + \\
&6\mathcal{H}\varphi_0'\phi_1^2 + 2\phi_1\varphi_0'\nabla^2 B_1 + 2\delta^{ij}\varphi_0'\partial_i B_1\partial_j\phi_1 - 2\delta^{ij}\varphi_0'\partial_i B_1\partial_j B_1' - 4\mathcal{H}\delta^{ij}\partial_i B_1\partial_j\varphi_1 - \\
&\delta^{ij}\partial_i B_1\partial_j B_1\varphi_0'' - 2\mathcal{H}\delta^{ij}\varphi_0'\partial_i B_1\partial_j B_1
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
C_{XX} =&a^{-2}\phi_2'{\varphi_0'}^3 - 3a^{-2}\varphi_0''{\varphi_1'}^2 - a^{-2}\varphi_2''{\varphi_0'}^2 - 12a^{-2}\phi_1\phi_1'{\varphi_0'}^3 + 8a^{-2}\phi_1\varphi_1''{\varphi_0'}^2 + \\
&4a^{-2}\phi_2\varphi_0''{\varphi_0'}^2 + 8a^{-2}\phi_1'\varphi_1'{\varphi_0'}^2 - 6a^{-2}\varphi_0'\varphi_1'\varphi_1'' - 3a^{-2}\varphi_0'\varphi_2'\varphi_0'' - 21a^{-2}\varphi_0''\phi_1^2{\varphi_0'}^2 - \\
&2a^{-2}\phi_1\nabla^2 B_1{\varphi_0'}^3 - 2a^{-2}\phi_1\nabla^2\varphi_1{\varphi_0'}^2 + 18a^{-2}\phi_1\varphi_0'\varphi_1'\varphi_0'' - a^{-2}\phi_2\mathcal{H}{\varphi_0'}^3 + 2a^{-2}\varphi_0'\varphi_1'\nabla^2\varphi_1 + \\
&4a^{-2}\delta^{ij}\varphi_0\partial_i\varphi_1\partial_j\varphi_1' + 2a^{-2}\varphi_1'\nabla^2 B_1{\varphi_0'}^2 - 2a^{-2}\delta^{ij}\partial_i B_1\partial_j\phi_1{\varphi_0'}^3 + \\
&2a^{-2}\delta^{ij}\partial_i B_1\partial_j B_1'{\varphi_0'}^3 + 4a^{-2}\delta^{ij}\partial_i B_1\partial_j\varphi_1'{\varphi_0'}^2 - 2a^{-2}\delta^{ij}\partial_i\phi_1\partial_j\varphi_1{\varphi_0'}^2 + a^{-2}\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1\varphi_0'' + \\
&2a^{-2}\delta^{ij}\partial_i\varphi_1\partial_j B_1'{\varphi_0'}^2 + a^{-2}\mathcal{H}\varphi_2'{\varphi_0'}^2 + 3a^{-2}\mathcal{H}\phi_1^2{\varphi_0'}^3 - 2a^{-2}\mathcal{H}\phi_1\varphi_1'{\varphi_0'}^2 + \\
&6a^{-2}\delta^{ij}\varphi_0'\partial_i B_1\partial_j\varphi_1\varphi_0'' + 4a^{-2}\delta^{ij}\partial_i B_1\partial_j B_1\varphi_0''{\varphi_0'}^2 - a^{-2}\mathcal{H}\delta^{ij}\partial_i B_1\partial_j B_1{\varphi_0'}^3 - \\
&2a^{-2}\mathcal{H}\delta^{ij}\partial_i B_1\partial_j\varphi_1{\varphi_0'}^2
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
C_{XXX} =& 2a^{-4}\phi_1\phi_1'{\varphi_0'}^5 - 2a^{-4}\phi_1\varphi_1''{\varphi_0'}^4 + a^{-4}\mathcal{H}\phi_2{\varphi_0'}^5 - a^{-4}\phi_2\varphi_0''{\varphi_0'}^4 - 2a^{-4}\phi_1'\varphi_1'{\varphi_0'}^4 + \\
& 2a^{-4}\varphi_1'\varphi_1''{\varphi_0'}^3 - a^{-4}\mathcal{H}\varphi_2'{\varphi_0'}^4 + a^{-4}\varphi_2'\varphi_0''{\varphi_0'}^3 - 8a^{-4}\mathcal{H}\phi_1^2{\varphi_0'}^5 - 5a^{-4}\mathcal{H}{\varphi_0'}^3{\varphi_1'}^2 + \\
& 11a^{-4}\varphi_0''\phi_1^2{\varphi_0'}^4 + 6a^{-4}\varphi_0''{\varphi_0'}^2{\varphi_1'}^2 + 12a^{-4}\mathcal{H}\phi_1\varphi_1'{\varphi_0'}^4 - 16a^{-4}\phi_1\varphi_1'\varphi_0''{\varphi_0'}^3 + \\
& a^{-4}\mathcal{H}\delta^{ij}\partial_i B_1\partial_j B_1{\varphi_0'}^5 + 2a^{-4}\mathcal{H}\delta^{ij}\partial_i B_1\partial_j\varphi_1{\varphi_0'}^4 + a^{-4}\mathcal{H}\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1{\varphi_0'}^3 - \\
& a^{-4}\delta^{ij}\partial_i B_1\partial_j B_1\varphi_0''{\varphi_0'}^4 - 2a^{-4}\delta^{ij}\partial_i B_1\partial_j\varphi_1\varphi_0''{\varphi_0'}^3 - a^{-4}\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1\varphi_0''{\varphi_0'}^2
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
C_{XXXX} =& a^{-6}\mathcal{H}\phi_1^2{\varphi_0'}^7 + a^{-6}\mathcal{H}{\varphi_0'}^5{\varphi_1'}^2 - a^{-6}\varphi_0''\phi_1^2{\varphi_0'}^6 - a^{-6}\varphi_0''{\varphi_0'}^4{\varphi_1'}^2 - \\
& 2a^{-6}\mathcal{H}\phi_1\varphi_1'{\varphi_0'}^6 + 2a^{-6}\phi_1\varphi_1'\varphi_0''{\varphi_0'}^5
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
C_{XXX\varphi} =& a^{-4}\phi_1^2{\varphi_0'}^6 + a^{-4}{\varphi_0'}^4{\varphi_1'}^2 - 2a^{-4}\phi_1\varphi_1'{\varphi_0'}^5 + 2a^{-4}\mathcal{H}\phi_1{\varphi_0'}^5\varphi_1 - 2a^{-4}\phi_1\varphi_0''{\varphi_0'}^4\varphi_1 - \\
& 2a^{-4}\mathcal{H}\varphi_1'{\varphi_0'}^4\varphi_1 + 2a^{-4}\varphi_1'\varphi_0''{\varphi_0'}^3\varphi_1
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
C_{XX\varphi} =& a^{-2}\phi_2{\varphi_0'}^4 - a^{-2}\varphi_2''{\varphi_0'}^3 - 5a^{-2}\phi_1^2{\varphi_0'}^4 - 4a^{-2}{\varphi_0'}^2{\varphi_1'}^2 + 8a^{-2}\phi_1\varphi_1'{\varphi_0'}^3 + \\
& 2a^{-2}\phi_1'{\varphi_0'}^3\varphi_1 - a^{-2}\varphi_0''{\varphi_0'}^2\varphi_2 - 2a^{-2}\varphi_1''{\varphi_0'}^2\varphi_1 + 8a^{-2}\phi_1\varphi_0''{\varphi_0'}^2\varphi_1 + \\
& a^{-2}\delta^{ij}\partial_i B_1\partial_j B_1{\varphi_0'}^4 + 2a^{-2}\delta^{ij}\partial_i B_1\partial_j\varphi_1{\varphi_0'}^3 + a^{-2}\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1{\varphi_0'}^2 - 6a^{-2}\varphi_0'\varphi_1'\varphi_0''\varphi_1 + \\
& a^{-2}\mathcal{H}{\varphi_0'}^3\varphi_2 - 2a^{-2}\mathcal{H}\phi_1{\varphi_0'}^3\varphi_1 + 2a^{-2}\mathcal{H}\varphi_1'{\varphi_0'}^2\varphi_1
\end{aligned}
\tag{A.6}
$$

$$
C_{XX\varphi\varphi} = 2a^{-2}\phi_1{\varphi_0'}^4\varphi_1 - 2a^{-2}\varphi_1'{\varphi_0'}^3\varphi_1 - a^{-2}\varphi_0''{\varphi_0'}^2\varphi_1^2 + a^{-2}\mathcal{H}{\varphi_0'}^3\varphi_1^2 \tag{A.7}
$$

$$
\begin{aligned}
C_{X\varphi} =& {\varphi_1'}^2 + \varphi_0'\varphi_2' + \varphi_0''\varphi_2 + 2\varphi_1''\varphi_1 + \phi_1^2{\varphi_0'}^2 - 2\phi_1\varphi_0'\varphi_1' - 2\phi_1\varphi_0''\varphi_1 - \\
& \delta^{ij}\partial_i\varphi_1\partial_j\varphi_1 - 2\delta^{ij}\partial_{ij}\varphi_1\varphi_1 - 2\phi_1'\varphi_0'\varphi_1 + 2\mathcal{H}\varphi_0'\varphi_2 + 4\mathcal{H}\varphi_1'\varphi_1 - \\
& 4\mathcal{H}\phi_1\varphi_0'\varphi_1 - 2\delta^{ij}\varphi_0'\partial_i B_1\partial_j\varphi_1 - 2\delta^{ij}\varphi_0'\partial_{ij}B_1\varphi_1
\end{aligned}
\tag{A.8}
$$

$$
C_{X\varphi\varphi} = \varphi_0''\varphi_1^2 + {\varphi_0'}^2\varphi_2 + 2\varphi_0'\varphi_1'\varphi_1 + 2\mathcal{H}\varphi_0'\varphi_1^2 \tag{A.9}
$$

$$
C_{X\varphi\varphi\varphi} = {\varphi_0'}^2\varphi_1^2 \tag{A.10}
$$

$$
C_{\varphi} = a^2\phi_2 - a^2\phi_1^2 + a^2\delta^{ij}\partial_i B_1\partial_j B_1 \tag{A.11}
$$

$$
C_{\varphi\varphi} = a^2\varphi_2 + 2a^2\phi_1\varphi_1 \tag{A.12}
$$

$$
C_{\varphi\varphi\varphi} = a^2\varphi_1^2 \tag{A.13}
$$

B Second order single variable equation of motion for non-canonical scalar field for Power-law inflation

Fourth order action for non-canonical scalar field, after partial integration is,

$$
\begin{aligned}
{}^{(4)}\mathcal{S} = \int d^4x \Big(& 3\phi_1\phi_2\delta^{ij}a'\partial_{ij}B_1\kappa^{-1}a + \frac{1}{8}P_X\phi_2^2{\varphi_0'}^2a^2 + \frac{1}{4}P\phi_2^2a^4 + \frac{1}{4}P_X\phi_2\delta^{ij}\partial_iB_1\partial_jB_1{\varphi_0'}^2a^2 + \frac{1}{2}P\phi_2\delta^{ij}\partial_iB_1\partial_jB_1a^4 - \\
& \frac{3}{2}P_X\phi_2\phi_1^2{\varphi_0'}^2a^2 - 3P\phi_2\phi_1^2a^4 - \frac{1}{4}\phi_2\delta^{ij}\delta^{kl}\partial_{ik}B_1\partial_{jl}B_1\kappa^{-1}a^2 + \frac{1}{4}\phi_2\delta^{ij}\delta^{kl}\partial_{ij}B_1\partial_{kl}B_1\kappa^{-1}a^2 - \frac{3}{2}P_X\phi_1\phi_2\varphi_0'\varphi_1'a^2 + \\
& \frac{1}{4}P_X\phi_2\varphi_0'\varphi_2'a^2 + \frac{1}{4}P_X\delta^{ij}\varphi_0'\varphi_2'\partial_iB_1\partial_jB_1a^2 - \frac{3}{4}P_X\varphi_0'\varphi_2'\phi_1^2a^2 + \frac{1}{4}P_X\phi_2{\varphi_1'}^2a^2 + \frac{1}{2}P_X\phi_1\varphi_1'\varphi_2'a^2 - \frac{1}{8}P_X{\varphi_2'}^2a^2 - \\
& \frac{1}{2}P_X\phi_2\delta^{ij}\varphi_0'\partial_iB_1\partial_j\varphi_1a^2 - \frac{1}{2}P_X\phi_1\delta^{ij}\varphi_0'\partial_iB_1\partial_j\varphi_2a^2 + \frac{1}{2}P_X\delta^{ij}\varphi_1'\partial_iB_1\partial_j\varphi_2a^2 + \frac{1}{2}P_X\delta^{ij}\varphi_2'\partial_iB_1\partial_j\varphi_1a^2 + \\
& \frac{1}{8}P_X\delta^{ij}\partial_i\varphi_2\partial_j\varphi_2a^2 - \frac{9}{4}P_{XX}\phi_2\phi_1^2{\varphi_0'}^4 + 3P_{XX}\phi_1\phi_2\varphi_1'{\varphi_0'}^3 + \frac{3}{2}P_{XX}\varphi_2'\phi_1^2{\varphi_0'}^3 - 2P_{XX}\phi_1\varphi_1'\varphi_2'{\varphi_0'}^2 + \\
& \frac{1}{2}P_{XX}\phi_1\delta^{ij}\partial_iB_1\partial_j\varphi_2{\varphi_0'}^3 + \frac{1}{2}P_{XX}\phi_1\delta^{ij}\partial_i\varphi_1\partial_j\varphi_2{\varphi_0'}^2 + \frac{1}{8}P_{XX}\phi_2^2{\varphi_0'}^4 + \frac{1}{4}P_{XX}\phi_2\delta^{ij}\partial_iB_1\partial_jB_1{\varphi_0'}^4 - \frac{1}{4}P_{XX}\phi_2\varphi_2'{\varphi_0'}^3 - \\
& P_{XX}\phi_2{\varphi_0'}^2{\varphi_1'}^2 + \frac{1}{2}P_{XX}\phi_2\delta^{ij}\partial_iB_1\partial_j\varphi_1{\varphi_0'}^3 + \frac{1}{4}P_{XX}\phi_2\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1{\varphi_0'}^2 - \frac{1}{4}P_{XX}\delta^{ij}\varphi_2'\partial_iB_1\partial_jB_1{\varphi_0'}^3 + \\
& \frac{3}{4}P_{XX}\varphi_0'\varphi_2'{\varphi_1'}^2 - \frac{1}{2}P_{XX}\delta^{ij}\varphi_1'\partial_iB_1\partial_j\varphi_2{\varphi_0'}^2 - \frac{1}{2}P_{XX}\delta^{ij}\varphi_0'\varphi_1'\partial_i\varphi_1\partial_j\varphi_2 + \frac{1}{8}P_{XX}{\varphi_0'}^2{\varphi_2'}^2 - \frac{1}{2}P_{XX}\delta^{ij}\varphi_2'\partial_iB_1\partial_j\varphi_1{\varphi_0'}^2 \\
& - \frac{1}{4}P_{XX}\delta^{ij}\varphi_0'\varphi_2'\partial_i\varphi_1\partial_j\varphi_1 + \frac{1}{8}P_{YY}\varphi_2^2a^4 + \frac{1}{4}P_{XY}\phi_2{\varphi_0'}^2a^2\varphi_2 + \frac{1}{4}P_{XY}\delta^{ij}\partial_iB_1\partial_jB_1{\varphi_0'}^2a^2\varphi_2 - \frac{1}{2}P_{XY}\phi_1^2{\varphi_0'}^2a^2\varphi_2 \\
& - P_{XY}\phi_1\phi_2{\varphi_0'}^2a^2\varphi_1 + \frac{1}{2}P_{XY}\phi_1\varphi_0'\varphi_1'a^2\varphi_2 + \frac{1}{2}P_{XY}\phi_2\varphi_0'\varphi_1'a^2\varphi_1 + \frac{1}{2}P_{XY}\phi_1\varphi_0'\varphi_2'a^2\varphi_1 - \frac{1}{4}P_{XY}\varphi_0'\varphi_2'a^2\varphi_2 - \\
& \frac{1}{4}P_{XY}{\varphi_1'}^2a^2\varphi_2 - \frac{1}{2}P_{XY}\varphi_1'\varphi_2'a^2\varphi_1 + \frac{1}{2}P_{XY}\delta^{ij}\varphi_0'\partial_iB_1\partial_j\varphi_1a^2\varphi_2 + \frac{1}{2}P_{XY}\delta^{ij}\varphi_0'\partial_iB_1\partial_j\varphi_2a^2\varphi_1 + \frac{1}{4}P_{XY}\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1a^2\varphi_2 + \\
& \frac{1}{2}P_{XY}\delta^{ij}\partial_i\varphi_1\partial_j\varphi_2a^2\varphi_1 + \frac{1}{4}P_{XXX}\phi_2\phi_1^2{\varphi_0'}^6a^{(-2)} - \frac{1}{4}P_{XXX}\varphi_2'\phi_1^2{\varphi_0'}^5a^{(-2)} - \frac{1}{2}P_{XXX}\phi_1\phi_2\varphi_1'{\varphi_0'}^5a^{(-2)} + \\
& \frac{1}{2}P_{XXX}\phi_1\varphi_1'\varphi_2'{\varphi_0'}^4a^{(-2)} + \frac{1}{4}P_{XXX}\phi_2{\varphi_0'}^4{\varphi_1'}^2a^{(-2)} - \frac{1}{4}P_{XXX}\varphi_2'{\varphi_0'}^3{\varphi_1'}^2a^{(-2)} + \frac{1}{4}P_{YYY}\varphi_1^2a^4\varphi_2 + \frac{1}{4}P_{XXY}\phi_1^2{\varphi_0'}^4\varphi_2 + \\
& \frac{1}{2}P_{XXY}\phi_1\phi_2{\varphi_0'}^4\varphi_1 - \frac{1}{2}P_{XXY}\phi_1\varphi_1'{\varphi_0'}^3\varphi_2 - \frac{1}{2}P_{XXY}\phi_1\varphi_2'{\varphi_0'}^3\varphi_1 - \frac{1}{2}P_{XXY}\phi_2\varphi_1'{\varphi_0'}^3\varphi_1 + \frac{1}{4}P_{XXY}{\varphi_0'}^2{\varphi_1'}^2\varphi_2 + \\
& \frac{1}{2}P_{XXY}\varphi_1'\varphi_2'{\varphi_0'}^2\varphi_1 + \frac{1}{2}P_{XYY}\phi_1{\varphi_0'}^2a^2\varphi_1\varphi_2 + \frac{1}{4}P_{XYY}\phi_2{\varphi_0'}^2\varphi_1^2a^2 - \frac{1}{2}P_{XYY}\varphi_0'\varphi_1'a^2\varphi_1\varphi_2 - \frac{1}{4}P_{XYY}\varphi_0'\varphi_2'\varphi_1^2a^2 + \\
& \frac{1}{2}P_X\phi_1\delta^{ij}\partial_i\varphi_1\partial_j\varphi_2a^2 + \frac{1}{2}P_{YY}\phi_1a^4\varphi_1\varphi_2 + \frac{1}{4}P_X\phi_2\delta^{ij}\partial_i\varphi_1\partial_j\varphi_1a^2 + \frac{1}{4}P_Y\phi_2a^4\varphi_2 + \frac{1}{4}P_{YY}\phi_2\varphi_1^2a^4 + \\
& \frac{1}{4}P_Y\delta^{ij}\partial_iB_1\partial_jB_1a^4\varphi_2 - \frac{1}{4}P_Y\phi_1^2a^4\varphi_2 - \frac{1}{2}P_Y\phi_1\phi_2a^4\varphi_1\Big)
\end{aligned}
\tag{B.1}
$$

We can eliminate ϕ_2, ϕ_1 and B_1 using constraint equations. While this is possible for canonical scalar field, it is highly non-trivial for non-canonical scalar fields. Hence we consider Power law inflation to reduce the action in a single variable form.

For Power-law, we have

$$a = a_0(-\eta)^\beta \tag{B.2}$$

$$\mathcal{H} \equiv \frac{a'}{a} = -\frac{\beta}{(-\eta)} \tag{B.3}$$

$$\frac{a''}{a} = -\frac{\beta(1-\beta)}{(-\eta)^2} \tag{B.4}$$

$$\varphi_0 = \varphi_c(-\eta)^\alpha \tag{B.5}$$

$${\varphi_0}' = \varphi_c\alpha(-\eta)^\alpha \tag{B.6}$$

$$\mathcal{H}^2 = -\frac{\kappa}{3}(P_X {\varphi_0'}^2 + P_0 a^2) \tag{B.7}$$

$$-2\frac{a''}{a} + \mathcal{H}^2 = \kappa P_0 a^2 \tag{B.8}$$

For Power-law inflation, $P(X,\varphi) = \frac{C}{\varphi^2}\ (X^2 + X)$. Solving for C, φ_c and α using above equations,

$$\alpha = 1 + \beta \tag{B.9}$$

$$C = \frac{-6\beta(1-\beta)}{\kappa(1+\beta)^2} \tag{B.10}$$

$$\varphi_{0c} = \sqrt{\frac{2}{3}}\frac{a_0}{1+\beta}\sqrt{\frac{1-2\beta}{1-\beta}} \tag{B.11}$$

Using first order Einstein's equations,

$$\phi_1 = \sqrt{\frac{3}{2}}\sqrt{\frac{1-\beta}{1-2\beta}}\frac{1+\beta}{a_0(-\eta)^{1+\beta}}\varphi_1 \tag{B.12}$$

$$\phi_1 = F_3\varphi_1 \tag{B.13}$$

$$\nabla^2 B_1 = \frac{3}{a_0}\sqrt{\frac{3}{2}}\sqrt{\frac{1-\beta}{1-2\beta}}(1-3\beta)\left\{\frac{1+\beta}{(-\eta)^{2+\beta}}\varphi_1 + \frac{1}{(-\eta)^{1+\beta}}\varphi_1'\right\} \tag{B.14}$$

$$\nabla^2 B_1 = F_1\varphi_1 + F_2\varphi_1' \tag{B.15}$$

where,

$$F_1 = \frac{3}{a_0}\sqrt{\frac{3}{2}}\sqrt{\frac{1-\beta}{1-2\beta}}(1-3\beta)\frac{1+\beta}{(-\eta)^{2+\beta}} \tag{B.16}$$

$$F_2 = \frac{3}{a_0}\sqrt{\frac{3}{2}}\sqrt{\frac{1-\beta}{1-2\beta}}(1-3\beta)\frac{1}{(-\eta)^{1+\beta}} \tag{B.17}$$

$$F_3 = \frac{1}{a_0}\sqrt{\frac{3}{2}}\sqrt{\frac{1-\beta}{1-2\beta}}\frac{1+\beta}{(-\eta)^{1+\beta}} \tag{B.18}$$

Similarly using second order Einstein's equations,

$$\partial_i\phi_2 = C_1\partial_i\varphi_2 + C_2\varphi_1\partial_i\varphi_1 + C_3\delta^{jk}\partial_j\varphi_1\partial_{ik}B_1 + C_4\delta^{jk}\partial_j B_1\partial_{ik}B_1 + C_5\varphi_1'\partial_i\varphi_1 \tag{B.19}$$

$$\phi_2 = C_1\varphi_2 + \frac{1}{2}C_2\varphi_1^2 - \delta^{ij}\partial_i B_1\partial_j B_1 + Q \tag{B.20}$$

$$Q = \partial^{-1i}\{C_3\delta^{jk}\partial_j\varphi_1\partial_{ik}B_1 + C_5\varphi_1'\partial_i\varphi_1\} \tag{B.21}$$

where

$$C_1 = \sqrt{\frac{3}{2}}\sqrt{\frac{1-\beta}{1-2\beta}}\frac{1+\beta}{a_0\ (-\eta)^{1+\beta}} \tag{B.22}$$

$$C_2 = -\frac{3}{2a_0^2}\frac{(1+\beta)(1-\beta)(3-14\beta+7\beta^2)}{(1-2\beta)}\frac{1}{(-\eta)^{2+2\beta}} \tag{B.23}$$

$$C_3 = \sqrt{\frac{3}{2}}\sqrt{\frac{1-\beta}{1-2\beta}}\frac{(1+\beta)}{a_0\beta}\frac{1}{(-\eta)^{\beta}} \tag{B.24}$$

$$C_4 = -2 \tag{B.25}$$

$$C_5 = -\frac{9}{2a_0^2}\frac{(1-\beta)^2(1-3\beta)}{\beta(1-2\beta)}\frac{1}{(-\eta)^{1+2\beta}} \tag{B.26}$$

Multiplying the fourth order action (B.1) by $\frac{8}{3}(1-2\beta)^2$,

$$\begin{aligned}&\int d^4x\Big[\Big(A_1\varphi_2'^2 + A_2\partial^i\varphi_2\partial_i\varphi_2 + A_3\varphi_2\varphi_2' + A_4\varphi_2^2\Big) + \partial^i\varphi_2\ \Big(B_1\varphi_1\partial_i\varphi_1 + B_2\varphi_1\partial_i B_1 + \\ &B_3\varphi_1'\partial_i B_1 + B_4\varphi_1'\partial_i\varphi_1\Big) + \varphi_2'\ \Big(D_1\varphi_1^2 + D_2\varphi_1'\varphi_1 + D_3 Q + D_4{\varphi_1'}^2 + D_5\partial^i B_1\partial_i\varphi_1 + \\ &D_6\partial^i\varphi_1\partial_i\varphi_1\Big) + \varphi_2\ \Big(E_1\varphi_1^2 + E_2(\nabla^2 B_1)^2 + E_3\partial^{ij}B_1\partial_{ij}B_1 + E_4\varphi_1\varphi_1' + E_5{\varphi_1'}^2 + E_6 Q \\ &+ E_7\partial^i B_1\partial_i\varphi_1 + E_8\partial^i\varphi_1\partial_i\varphi_1\Big)\Big]\end{aligned} \tag{B.27}$$

where,

$$A_1 = -\frac{3\beta(1-\beta)(1-2\beta)(1-3\beta)}{\kappa(-\eta)^2} \tag{B.28}$$

$$A_2 = -\frac{\beta(1-\beta)(1+\beta)(1-2\beta)}{\kappa(-\eta)^2} \tag{B.29}$$

$$A_3 = -\frac{2\beta(1-\beta)(1+\beta)(1-2\beta)(1-11\beta)}{\kappa(-\eta)^3} \tag{B.30}$$

$$A_4 = \frac{3\beta(1-\beta)(1+\beta)^2(1-2\beta)(1+3\beta)}{\kappa(-\eta)^4} \tag{B.31}$$

$$B_1 = -\frac{36\beta(1-\beta)(1+\beta)(1-2\beta)(1-3\beta)}{\sqrt{6}a_0\ \kappa(-\eta)^{3+\beta}}\sqrt{\frac{1-\beta}{1-2\beta}} \tag{B.32}$$

$$B_2 = \frac{4\beta(1-\beta)(1+\beta)(1-2\beta)(1-11\beta)}{\kappa(-\eta)^3} \tag{B.33}$$

$$B_3 = \frac{12\beta(1-\beta)(1-2\beta)(1-3\beta)}{\kappa(-\eta)^2} \tag{B.34}$$

$$B_4 = -\frac{48\beta(1-2\beta)^2(1-\beta)}{\sqrt{6}\ a_0\ \kappa(-\eta)^{2+\beta}}\sqrt{\frac{1-\beta}{1-2\beta}} \tag{B.35}$$

$$D_1 = \frac{9\sqrt{(1-\beta)(1-2\beta)}(1+\beta)(1-\beta)(3-5\beta+5\beta^2-83\beta^3)}{\sqrt{6}\ a_0\ \kappa(-\eta)^{4+\beta}} \tag{B.36}$$

$$D_2 = \frac{36\beta\sqrt{(1-\beta)(1-2\beta)}(1-\beta)(1+\beta)(7-17\beta)}{\sqrt{6}\ a_0\ \kappa(-\eta)^{3+\beta}} \tag{B.37}$$

$$D_3 = -\frac{-12\ a_0\ \beta\sqrt{(1-\beta)(1-2\beta)}(1-2\beta)(1-3\beta)}{\sqrt{6}\kappa(-\eta)^{2-\beta}} \tag{B.38}$$

$$D_4 = \frac{72\beta\sqrt{(1-\beta)(1-2\beta)}(1-\beta)(1-2\beta)}{\sqrt{6}\ a_0\ \kappa(-\eta)^{2+\beta}} \tag{B.39}$$

$$D_5 = \frac{12\beta(1-\beta)(1-2\beta)(1-3\beta)}{\kappa(-\eta)^2} \tag{B.40}$$

$$D_6 = -\frac{24\beta\sqrt{(1-2\beta)(1-\beta)}(1-2\beta)(1-\beta)}{\sqrt{6}\ a_0\ \kappa(-\eta)^{2+\beta}} \tag{B.41}$$

$$E_1 = \frac{9\sqrt{(1-\beta)(1-2\beta)}(1-\beta)(1+\beta)^2(3-19\beta-3\beta^2-77\beta^3)}{\sqrt{6}\ a_0\ \kappa(-\eta)^{5+\beta}} \tag{B.42}$$

$$E_2 = \frac{2\ a_0\ \sqrt{(1-\beta)(1-2\beta)}(1+\beta)(1-2\beta)}{\sqrt{6}\ \kappa(-\eta)^{1-\beta}} \tag{B.43}$$

$$E_3 = -\frac{2\ a_0\ \sqrt{(1-\beta)(1-2\beta)}(1+\beta)(1-2\beta)}{\sqrt{6}\ \kappa(-\eta)^{1-\beta}} \tag{B.44}$$

$$E_4 = \frac{72\beta\sqrt{(1-\beta)(1-2\beta)}(1-\beta)^2(1+\beta)(3+11\beta)}{\sqrt{6}\ a_0\ \kappa(-\eta)^{4+\beta}} \tag{B.45}$$

$$E_5 = \frac{18\beta\sqrt{(1-\beta)(1-2\beta)}(1-\beta)(1+\beta)(7-17\beta)}{\sqrt{6}\ a_0\ \kappa(-\eta)^{3+\beta}} \tag{B.46}$$

$$E_6 = -\frac{12\ a_0\ \beta\sqrt{(1-\beta)(1-2\beta)}(1-2\beta)(1+\beta)(1-3\beta)}{\sqrt{6}\ \kappa(-\eta)^{3+\beta}} \tag{B.47}$$

$$E_7 = \frac{4\beta(1-\beta)(1-2\beta)(1+\beta)(1-11\beta)}{\kappa(-\eta)^3} \tag{B.48}$$

$$E_8 = -\frac{18\beta\sqrt{(1-\beta)(1-2\beta)}(1-\beta)(1+\beta)(1-3\beta)}{\sqrt{6}\ a_0\ \kappa(-\eta)^{3+\beta}} \tag{B.49}$$

$$\tag{B.50}$$

After taking partial derivative,

$$\int d^4x\Big[\Big(A_1\varphi_2'^2 + A_2\partial^i\varphi_2\partial_i\varphi_2 + A_5\varphi_2^2\Big) + \varphi_2\ \Big(G_1\varphi_1^2 + E_2(\nabla^2 B_1)^2 + E_3\partial^{ij}B_1\partial_{ij}B_1 +$$
$$G_2\varphi_1\varphi_1' + G_3{\varphi_1'}^2 + G_4\varphi_1\varphi_1'' + G_5\varphi_1'\varphi_1'' + G_6\varphi\nabla^2\varphi_1 + G_7\varphi'\nabla^2\varphi_1 +$$
$$G_8\partial^i\varphi_1\partial_i\varphi_1 + G_9\partial^i\varphi_1'\partial_i\varphi_1 + G_{10}\partial^i\varphi_1\partial_i B_1 + G_{11}\partial^i\varphi_1'\partial_i B_1 + G_{12}Q + G_{13}Q'\Big)\Big] \tag{B.51}$$

where,

$$G_1 = E_1 - B_2F_1 - D_1' \tag{B.52}$$
$$G_2 = E_4 - (B_2F_2 + B_3F_1) - (D_2' + 2D_1) \tag{B.53}$$
$$G_3 = E_5 - B_3F_2 - (D_2 + D_4') \tag{B.54}$$
$$G_4 = -D_2 \tag{B.55}$$
$$G_5 = -2D_4 \tag{B.56}$$
$$G_6 = -B_1 \tag{B.57}$$
$$G_7 = -B_4 \tag{B.58}$$
$$G_8 = E_8 - B_1 - (D_6' - D_5F_3) \tag{B.59}$$
$$G_9 = -B_4 - 2D_6 \tag{B.60}$$
$$G_{10} = E_7 - B_2 - (D_5' - 2\mathcal{H}D_5) \tag{B.61}$$
$$G_{11} = -B_3 - D_5 \tag{B.62}$$
$$G_{12} = E_6 - D_3' \tag{B.63}$$
$$G_{13} = -D_3 \tag{B.64}$$

so the equation of motion of scalar field for power law K-Inflation is,

$$\begin{aligned}&2A_1\varphi_2'' + 2A_1'\varphi_2' + A_2\nabla^2\varphi_2 + A_5\varphi_2^2 = G_1\varphi_1^2 + E_2(\nabla^2B_1)^2 + E_3\partial^{ij}B_1\partial_{ij}B_1 +\\ &G_2\varphi_1\varphi_1' + G_3{\varphi_1'}^2 + G_4\varphi_1\varphi_1'' + G_5\varphi_1'\varphi_1'' + G_6\varphi\nabla^2\varphi_1 + G_7\varphi'\nabla^2\varphi_1 +\\ &G_8\partial^i\varphi_1\partial_i\varphi_1 + G_9\partial^i\varphi_1'\partial_i\varphi_1 + G_{10}\partial^i\varphi_1\partial_iB_1 + G_{11}\partial^i\varphi_1'\partial_iB_1 + G_{12}Q + G_{13}Q'\end{aligned} \tag{B.65}$$

a_0 dependency in the action as well as in the equation of motion can be resolved by rescaling $\varphi_1 \to a_0\varphi_1, \varphi_2 \to a_0\varphi_2$.

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